QFI QF Model Solutions
Fall 2022

1. **Learning Objectives:**
1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:
(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:
Hirsa and Neftci; An Introduction to the Mathematics of Financial Derivatives; Chapters 10, 11

Chin; Problems and Solutions Math Finance; Chapters 2, 3

Commentary on Question:
Most candidates did well on this problem.

Solution:
(a) List the criteria for a stochastic process to be a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Commentary on Question:
Most candidates were able to list the three criteria. However, some candidates forgot the absolute value when stating the second criterion.

The three criteria for $0 \leq s \leq t \leq T$ are:
$E^{\mathbb{P}}(V_t|\mathcal{F}_s) = V_s$;
The equality holds almost surely.
$E^{\mathbb{P}}[|V_t|] < \infty$;
And the 3rd criterion is that V_t is $\mathcal{F}_t$-measurable

(b) Derive a stochastic differential equation for X_t using Ito's Lemma.

Commentary on Question:
Most candidates did this part correctly. Some candidates did not use the correct notation that is specific to this problem.

1. Continued

By Ito's Lemma, one can derive

$$dX_t = W_t{}^2 dt + \frac{\partial \alpha_t}{\partial t} dt + \frac{\partial \alpha_t}{\partial W_t} dW_t + \frac{1}{2}\frac{\partial^2 \alpha_t}{\partial W_t^2} dt$$

(c) Identify an appropriate β_t, if it exists, that makes X_t a martingale.

Commentary on Question:
Many candidates were able to derive the correct function. Most candidates were able to state that a martingale requires the drift to be zero.

From part (b), it follows that $W_t{}^2 + \frac{\partial \alpha_t}{\partial t} + \frac{1}{2}\frac{\partial^2 \alpha_t}{\partial W_t^2} = 0 \qquad (*)$
implies that X_t is a martingale.

Since $\alpha_t = -tW_t{}^2 + \beta_t$ where β_t is deterministic, we have

$dX_t = (W_t{}^2 - W_t{}^2)dt + \frac{\partial \beta_t}{\partial t} dt - 2\mathrm{t}W_t dW_t - tdt.$

Therefore, by letting $\beta_t = \frac{1}{2}\mathrm{t}^2$ and $\alpha_t = -tW_t{}^2 + \frac{1}{2}\mathrm{t}^2$, X_t becomes a martingale.

(d) Calculate $E(W_t{}^4)$ using Ito's Lemma.

Commentary on Question:
Most candidates did this part correctly.

We use Ito's Lemma on $f(W_t) = W_t{}^4$.

$$dW_t{}^4 = 4W_t{}^3 dW_t + 6W_t{}^2 dt$$

therefore

$$W_t{}^4 = 4\int_0^t W_s{}^3 dW_s + 6\int_0^t W_s{}^2 ds$$

We then obtain

$$E(W_t{}^4) = 6E\left(\int_0^t W_s{}^2 ds\right) \qquad (**).$$

One can evaluate $(**)$ as follows (By Fubini's theorem):

$$E\left(\int_0^t W_s{}^2 ds\right) = \int_0^t E(W_s{}^2) ds = \int_0^t s ds = \frac{1}{2}t^2.$$

1. Continued

Alternatively, a candidate can use part (c) in the following manner:

Since X_t is a martingale,

$$E\left(\int_0^t W_s{}^2 ds\right) + E(\alpha_t) = 0 \Rightarrow E\left(\int_0^t W_s{}^2 ds\right) = E\left(tW_t{}^2 - \frac{1}{2}\mathrm{t}^2\right) = tE(W_t{}^2) - \frac{1}{2}\mathrm{t}^2 = \frac{1}{2}\mathrm{t}^2.$$

In either case, the result is:

$$E(W_t{}^4) = 3t^2.$$

2. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1b) Understand the importance of the no-arbitrage condition in asset pricing

(1f) Understand and apply Jensen's Inequality

Sources:

Hirsa and Neftci; An Introduction to the Mathematics of Financial Derivatives;
Wilmott; Frequently Asked Questions in Quantitative Finance

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Determine the range of α so that there is no arbitrage opportunity.

Commentary on Question:
Candidates did poorly on this part. Many failed to recognize the conditions needed to avoid arbitrage.

If one invests S_{t-1} in the risk-free asset at the beginning of year t, then the expected return is $1.05S_{t-1}$.

Therefore, the following inequality should be satisfied to avoid arbitrage:$\alpha S_{t-1} < 1.05S_{t-1} < 1.3S_{t-1}$ at $t = 1, 2, \ldots$

Thus, $\alpha < 1.05$.

(b) Derive the price S_4 as a function of d and give the possible range of S_4.

Commentary on Question:
Candidates did well on this part.

$S_4 = 100\alpha^{d} 1.3^{4-d}$.

Thus, $100\alpha^{4} < S_4 < 285.61$

(c) Calculate the real-world probability that the double barrier option will be exercised.

2. **Continued**

Commentary on Question:
Candidates did poorly on this part. Many incorrectly states that 5 down movements were needed to exercise the barrier option.

Since the upper barrier is 290, the barrier can be hit only if all 5 stock price movements are up.
In the same manner, the lower barrier 70 can be hit only if the first 4 movements are all down.

Therefore, the real-world probability of the double barrier option being exercized is:

$$(0.65)(0.6)(0.55)(0.5)^2 + (0.35)(0.4)(0.45)(0.5) = 8.5125\%.$$

(d) Calculate the risk-neutral probability of an up-movement in the price of the asset S.

Commentary on Question:
Candidates did well on this part.

Let us define the risk-neutral probability of up-movement as q.

Then, we can solve the following equation for q:
$1.05 = q \times 1.3 + (1 - q) \times 0.9.$

Thus, $q = \frac{15}{40} = 37.5\%.$

(e) Calculate the price Z_0 of the double barrier option at $t = 0$ under the risk-neutral measure.

Commentary on Question:
Candidates did well on this part. Candidates that incorrectly performed previous parts were not penalized for the same mistakes again.

The risk-neutral expected payoff of the option at time 5 is

$$E^{\mathrm{Q}}[Z] = 100(q^5 + (1 - q)^4) = 16.$$

The time-0 price of the option is:

$$Z_0 = \frac{E^{\mathrm{Q}}[Z]}{(1.05)^5} = 12.54.$$

2. Continued

(f) Derive a lower bound for the price C_0 of the call option at $t = 0$ under the risk-neutral measure using Jensen's inequality.

Commentary on Question:
Candidates performed poorly on this part. Most correctly stated Jensen's inequality but could not apply it to the price of a call.

One can easily verify that $f(x) = (x - 110)_+$ is a non-negative and convex function. Thus, we have
$C_0 = E^{\mathrm{Q}}[C] = (1.05)^{-5} E^{\mathrm{Q}}[f(S_5)] \geq (1.05)^{-5} f(E^{\mathrm{Q}}[S_5])$.

Since Q is the risk neutral measure, it is straightforward to see that
$E^{\mathrm{Q}}[S_5] = (1.05)^5 S_0 = 127.63$.

Therefore, $C_0 \geq (1.05)^{-5}[127.63 - 110] = 13.81$.

3. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1b) Understand the importance of the no-arbitrage condition in asset pricing.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1i) Demonstrate understanding of the differences and implications of real-world versus risk-neutral probability measures, and when the use of each is appropriate.

(1j) Understand and apply Girsanov's theorem in changing measures.

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

Sources:

Chin p. 156 Question 2, Neftci, QFIQ-113-17

Commentary on Question:

Overall, candidates performed as expected on this question. Parts b) and c) had the best responses. Few candidates scored well on part d) and found the manipulation in Girsanov's theorem a challenge. Parts a) and e) were effectively straight but candidates in general struggled, especially with part e) where it was clear that few understood the material.

Solution:

(a) List two potential issues of using real-world probabilities to price an EIA.

Commentary on Question:

This was straight recital from Pg 113 of QFIQ-113-17. Other answers were accepted provided they were reasonable

3. Continued

The two main problems associated with pricing with real probabilities are:

1. You need to be able to measure real probablities. This can be harder than measuring volatilities.
2. You need to decide on a utility function or a measure of risk aversion.

(b) Show that the discounted stock index process $e^{-\int_0^t r(u)du} S(t)$ is a martingale with respect to $\mathbb{Q}$.

Commentary on Question:
Nearly half of candidates answered this question successfully using Ito's Lemma as shown below. Some candidates attempted to use the definition of a Martingale, which in theory is perfectly acceptable; however, most candidates who used this approach failed to demonstrate the martingale property correctly. They struggled manipulating the conditional expectation and the properties of filtrations

Use Ito's Lemma:

$$d\left(e^{-\int_0^t r(u)du} S(t)\right) = -r(t)\, e^{-\int_0^t r(u)du} S(t)\, dt + e^{-\int_0^t r(u)du} dS(t)$$

$$= -r(t)\, e^{-\int_0^t r(u)du} S(t)\, dt + e^{-\int_0^t r(u)du}\, r(t)\, S(t)\, dt$$

$$+ e^{-\int_0^t r(u)du} \sigma_S(t)\, S(t)\, dW_S(t) = e^{-\int_0^t r(u)du} \sigma_S(t)\, S(t)\, dW_S(t).$$

which is driftless, hence the result.

(c) Show that for some deterministic function $\theta(t)$,

$$\frac{dF(t)}{F(t)} = \theta(t)dt + [\sigma_S(t)\, dW_S(t) - \sigma_B(t,T)\, dW_B(t)]$$

Commentary on Question:
Overall candidates performed well on this part. A couple struggled to successfully clean up and identify theta. A small minority did not attempt the question or did not know how to use multivariate Ito's lemma

3. **Continued**

$$
\begin{aligned}
dF(t) &= \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial B}dB + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 F}{\partial B^2}(dB)^2 + \frac{\partial^2 F}{\partial S \partial S}(dS)(dB) \\
&= \frac{1}{B(t,T)}\big(r(t)S(t)\,dt + \sigma_S(t)S(t)\,dW_S(t)\big) \\
&- \frac{S(t)}{B(t,T)^2}\big(r(t)B(t,T)dt + \sigma_B(t,T)B(t,T)\,dW_B(t)\big) \\
&+ \frac{S(t)}{B(t,T)^3}\sigma_B(t,T)^2B(t,T)^2dt - \frac{1}{B(t,T)^2}\rho\sigma_S(t)\sigma_B(t,T)S(t)B(t,T)dt \\
&= \frac{S(t)}{B(t,T)}[\sigma_S(t)\,dW_S(t) - \sigma_B(t,T)\,dW_B(t)] \\
&+ \frac{S(t)}{B(t,T)}[\sigma_B(t,T)^2 - \rho\sigma_S(t)\sigma_B(t,T)]dt.
\end{aligned}
$$

Rewrite as follows:

$$\frac{dF(t)}{F(t)} = [\sigma_B(t,T)^2 - \rho\sigma_S(t)\sigma_B(t,T)]dt + [\sigma_S(t)\,dW_S(t) - \sigma_B(t,T)\,dW_B(t)].$$

(d) Show that $\frac{dF(t)}{F(t)} = \sigma(t)\,dW^*(t)$ for suitably defined $\sigma(t), W^*(t)$, and probability measure $\mathbb{P}$ using Girsanov's theorem.

Commentary on Question:
Most candidates did not do well on this question.

Let $\sigma(t) = \sqrt{\sigma_S(t)^2 - 2\rho\sigma_S(t)\sigma_B(t,T) + \sigma_B(t,T)^2}$, then

$$\frac{dF(t)}{F(t)} = [\sigma_B(t)^2 - \rho\sigma_S(t)\sigma_B(t,T)]dt + \sigma(t)\,dW(t).$$

Define $\theta_t = \sigma_B(t,T)^2 - \rho\sigma_S(t)\sigma_B(t,T)$ and

$$\xi_t = e^{\int_0^t (\frac{\theta_u}{\sigma(u)})dW(u) - 1/2\int_0^t (\frac{\theta_u}{\sigma(u)})^2 du}$$

for $t > 0$. Then ξ_t is a martingale with respect to $\mathcal{F}_t$ and $\mathbb{Q}$, and

$$W^*(t) = \int_0^t \frac{\theta_u}{\sigma(u)}\,du + W(t)$$

3. Continued

is a Wiener process with respect to the same $\mathcal{F}_t$ and probability measure $\mathbb{P}$, which is defined by

$$\mathbb{P}(A) = E^{\mathbb{Q}}(\mathbb{1}_A \xi_t)$$

for any event A in $\mathcal{F}_t$. The conclusion is that

$$dW^*(t) = \frac{\theta_t}{\sigma(t)}\, dt + dW(t)$$

therefore

$$\frac{\mathrm{dF(t)}}{\mathrm{F(t)}} = \sigma(\mathrm{t})\ \mathrm{dW}^*(\mathrm{t}).$$

(e) Express C_t in terms of a risk-neutral expectation involving *r(t)* and *S(t).*

Commentary on Question:
Most candidates did not know the material asked in this question. Of those who attempted it, most completely excluded the expectation despite the question asking them to include one.

For the EIA this is equal to $\prod_{k=1}^{s} \max\left(1 + 0.8R(k), e^{0.01}\right)$
where R(k) = $\frac{\mathrm{S(k)}}{S(k-1)} - 1.$

$$C(t,s) = E^{\mathbb{P}}\left[e^{-\int_t^s r(u)du} \prod_{k=1}^{s} \max\left(1 + 0.8R(k), e^{0.01}\right) \mid F_t\right]$$

4. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 9)

Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014 (pages 52, 66,146)

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Show that $Z_t = \int_0^t W_u du$ is a normally distributed random variable.

Commentary on Question:

Most candidates were able to show that Z_t was normal; a full credit response required the use of the mean square limit as justification.

We can use integration by parts to show $d(tW_t) = tdW_t + W_t dt$. Therefore,

$$Z_t = \int_0^t W_u \, du = tW_t - \int_0^t u \, dW_u = \int_0^t (t-u) \, dW_u$$

The last integral is the mean square limit of $\sum_{k=1}^{n}(t-u_{k-1})\,(W_k - W_{k-1})$for a subdivision $0 = u_0 < u_1 < \cdots < u_n = t$ and $W_k := W_{u_k}$. This sum represents a linear combination of independent normally distributed random variables, which is also normal in the limit.

<u>Alternative solution:</u>

$Z_t = \int_0^t W_u \, du$ is the mean square limit of $\sum_{k=1}^{n} W_k(u_k - u_{k-1})$ for a subdivision $0 = u_0 < u_1 < \cdots < u_n = t$ and $W_k := W_{u_k}$. This sum represents a linear combination of independent normally distributed random variables, which is also normal in the limit.

4. Continued

(b) Determine whether Y_t is a Wiener process with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Commentary on Question:
Most candidates were able to derive the expectation and variance of Y_t correctly but were not able to identify the incremental variance as the reason it fails to be a Wiener process. Some candidates incorrectly deduced Y_t was a Wiener process based on the expectation and variance.

In order to be a Wiener process with respect to filtration $\{\mathcal{F}_t\}_{t\geq 0}$, Y_t should satisfy:

i. $Y_0 = 0$ and has continuous sample paths
ii. For $t > 0, s > 0$, $Y_{t+s} - Y_t \sim N(0, s)$ [stationary increment]
iii. For $t > 0, s > 0$, $Y_{t+s} - Y_t \perp Y_t$ [independent increment]

We start with the expectation and variance of Y_t:

$E[Y_t] = E\left[\frac{\sqrt{3}}{t} Z_t\right] = E\left[\frac{\sqrt{3}}{t}\int_0^t (t-u)\, dW_u\right] = \frac{\sqrt{3}}{t} E\left[\int_0^t (t-u)\, dW_u\right] = 0$, as it is an Ito integral.

$$Var[Y_t] = Var\left[\frac{\sqrt{3}}{t} Z_t\right] = E\left[\left(\frac{\sqrt{3}}{t} Z_t\right)^2\right] = \frac{3}{t^2} E\left[{Z_t}^2\right] = \frac{3}{t^2}\int_0^t (t-u)^2\, du = \frac{1}{t^2} t^3$$
$$= t$$

by Ito's isometry.

4. **Continued**

We can use the above to determine the incremental variance:

$$
\begin{aligned}
Var[Y_{t+s} - Y_t] &= Var\left[\frac{\sqrt{3}}{t+s} Z_{t+s} - \frac{\sqrt{3}}{t} Z_t\right] \\
&= Var\left[\frac{\sqrt{3}}{t+s} Z_{t+s}\right] + Var\left[\frac{\sqrt{3}}{t} Z_t\right] - 2Cov\left[\frac{\sqrt{3}}{t+s} Z_{t+s}, \frac{\sqrt{3}}{t} Z_t\right] \\
&= (t+s) + t \\
&- \frac{6}{t(t+s)} E\left[\int_0^{t+s} (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right] \\
&= 2t + s \\
&- \frac{6}{t(t+s)}\left[E\left[\int_t^{t+s} (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right]\right. \\
&\left.+ E\left[\int_0^t (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right]\right] \\
&= 2t + s - \frac{6}{t(t+s)} \int_0^t (t+s-u)\,(t-u) du \\
&= 2t + s - \frac{6}{t(t+s)}\left(\frac{st^2}{2} + \frac{t^3}{3}\right) = \frac{s^2}{t+s} \neq s
\end{aligned}
$$

So, Y_t fails to have the stationary increment property and is therefore not a Wiener process.

<u>Alternative solution:</u>
For a Wiener process, W_t, we expect $Cov[W_{t+s}, W_t\,] = t$. However,

$$
\begin{aligned}
Cov\left[\frac{\sqrt{3}}{t+s} Z_{t+s}, \frac{\sqrt{3}}{t} Z_t\right] &= \frac{3}{t(t+s)} E\left[\int_0^{t+s} (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right] \\
&= \frac{3}{t(t+s)}\left[E\left[\int_t^{t+s} (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right]\right. \\
&\left.+ E\left[\int_0^t (t+s-u)\, dW_u \int_0^t (t-u)\, dW_u\right]\right] \\
&= \frac{3}{t(t+s)} \int_0^t (t+s-u)\,(t-u) du = \frac{3}{t(t+s)}\left(\frac{st^2}{2} + \frac{t^3}{3}\right) \\
&= \frac{3st + 2t^2}{2(t+s)} = t + \frac{st}{2(t+s)} \neq t
\end{aligned}
$$

Therefore, Y_t is not a Wiener process.

4. Continued

(c) Derive an expression for dG_t in terms of dY_t.

Commentary on Question:
Few candidates performed well on this section. Most failed to establish any relationship between $\mathrm{G_t}$ *and* $\mathrm{Y_t}$.

We can easily derive an expression for $\ln S_u$, given the process satisfies the GBM model:

$$S_u = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)u + \sigma W_u} \Leftrightarrow \ln S_u = \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)u + \sigma W_u$$

We can substitute the above into the expression for $\ln G_t$:

$$\begin{aligned}
\ln G_t &= \frac{1}{t}\int_0^t \left(\ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)u + \sigma W_u\right) du \\
&= \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{t}{2} + \frac{\sigma}{t}\int_0^t W_u\, du \\
&= \ln S_0 + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{t}{2} + \frac{\sigma}{\sqrt{3}} Y_t
\end{aligned}$$

This allows us to re-write the process, G_t as:

$$G_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)\frac{t}{2} + \frac{\sigma}{\sqrt{3}} Y_t}$$

By applying Ito's Lemma to G_t, we find:

$$dG_t = G_t\left[\frac{1}{2}\left(\mu - \frac{1}{2}\sigma^2\right)dt + \frac{\sigma}{\sqrt{3}}dY_t + \frac{1}{2}\left(\frac{\sigma^2}{3}\right)dt\right],$$

or equivalently:

$$\frac{dG_t}{G_t} = \frac{1}{2}\left(\mu - \frac{1}{6}\sigma^2\right)dt + \frac{\sigma}{\sqrt{3}}dY_t$$

5. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Chs 5 – 6

Commentary on Question:

The questions tested candidates understanding of forwards and futures. In general, the candidates performed well, particularly when asked to discuss features and uses of interest rate instruments (parts d and e). Candidates performed below expectation when asked to calculate the value of a forward rate agreement (parts b and c).

Solution:

(a) Calculate the current 1 year forward rates implied by the above initial spot rates.

Commentary on Question:

The calculation was straightforward. A common error was to shift the forward rates by one period. Some candidates used a different formula.

$$f_n(t,T_1,T_2)*\Delta = \frac{Z(t,T_1)}{Z(t,T_2)} - 1$$

	Discount factor	Spot Rate	Forward Rate
0	1		
1	0.99009901	0.01	0.03009901
2	0.961168781	0.02	0.050295079
3	0.915141659	0.03	0.070586304
4	0.854804191	0.04	0.090970829

(b) Describe the cost to enter into a Forward Rate Agreement (FRA) today on $100 million for one year starting 3 years into the future assuming no transaction costs.

Commentary on Question:

Candidates performed below expectation on this question. Many candidates attempted to calculate a cost to enter the Forward Rate Agreement.

There is no cost associated with entering into a forward rate agreement. (Besides some transactions costs in the real world.)

5. Continued

(c) Determine the loss or gain if you were to exit the FRA at time 1.

Commentary on Question:
Candidates performed poorly on this question. Many candidates did not use the correct formula for the value of a forward rate agreement or were missing a part of the formula. A common mistake was that candidates used the forward rates and discount factors from the wrong time period.

$$V^{FRA}(t) = N * Z(t,T_4) * \Delta * \left[f_1(0,T_3,T_4) - f_1(t,T_3,T_4) \right]$$

	Discount Factor	Spot Rate	Forward Rate
0	1		
1	0.99009901	0.01	0.03009901
2	0.961168781	0.02	0.050795079
3	0.915141659	0.03	0.070586304
4	0.854804191	0.04	0.090970829
5	0.783526466	0.05	
1	1		
2	0.992555831	0.0075	0.027599256
3	0.965897772	0.0175	0.047795806
4	0.921837791	0.0275	0.06808774
5	0.863073095	0.0375	

$Z(1,T_4) = 0.9218$
$f_1(0,T_3,T_4) = 0.070586304$
$f_1(1,T_3,T_4) = 0.047795806$
$V^{FRA}(1) = 1000000 * Z(1,T_4) * [f_1(0,T_3,T_4) - f_1(0,T_3,T_4)]$
$= -2{,}100{,}914.1985$

(d) Discuss the advantages and disadvantages of using forwards rather than futures for each of the following risks:

(i) Basis Risk

(ii) Tailing of the hedge

(iii) Liquidity

(iv) Credit Risk

5. Continued

Commentary on Question:
Candidates performed very well on this question. Candidates needed to state which instrument had an advantage and provide an explanation.

Basis risk – Advantage for the forward, as futures may have maturity mismatch between the needs of the firm and the available maturities. Forward would be customized.
Tailing of the hedge- Advantage to a forward, as the cash flows from the future accrue over time (so time value of money must be taken into account.)
Liquidity - Advantage to the future. Forward contracts are harder to get out of the position as they are traded over the counter.
Credit Risk – Advantage to the future. Clearing house guarantees performance on futures contracts. Forward has risk with the counterparty.

(e) Describe one other instrument that may be more suitable to protect against such a situation.

Commentary on Question:
Candidates performed well on this question. A variety of instruments were acceptable solutions.

Interest Rate Put Options with parameters tied to the initial yield curve. Given that we are looking to emulate a forward contract, using a futures option on a 1-year note is likely a reasonable approach.

6. Learning Objectives:

2. The candidate will understand the fundamentals of fixed income markets and traded securities.

Learning Outcomes:

(2a) Understand the characteristics of fixed rate, floating rate, and zero-coupon bonds.

(2d) Understand the characteristics and uses of interest rate forwards, swaps, futures, and options.

Sources:

Fixed Income Securities (Chapter 5)

QFIQ-121-20: A Guide to Duration, DV01, and Yield Curve Risk Transformations

Commentary on Question:

This question tests candidates' understanding of the meaning of the dollar value of a basis point and the relationship of this metric with forward rates.

Solution:

(a) Express partial DV01s for *n* assets with respect to the forward rate variables in the Jacobian matrix multiplication form by perturbating the curve rather than perturbating the inputs.

Commentary on Question:

Most candidates wrote down the operational definition of the DV01 and about 3 out of 4 candidates wrote down the entire expression.

The partial DV01 (dollar value of one basis point) of a Portfolio, P, with respect to the vector of forward rates (f_1 , f_2 , f_3 , f_4, ...) is given by:

$$\frac{\partial P}{\partial f} = \frac{\partial P}{\partial r} \times \left(\frac{\partial f}{dr}\right)^{-1}$$

(b) Calculate the partial DV01 of the portfolio *P* with respect to forward rates (f_1,f_2,f_3,f_4) based on the four sets of perturbed discount factors.

Commentary on Question:

Most candidates did not attempt this question at all. Candidates that did attempt the question received credit for the first step in the process.

The value is calculated using the formula from part (a):

$$\frac{\partial P}{\partial f} = \frac{\partial P}{\partial r} \times \left(\frac{\partial f}{dr}\right)^{-1}$$

6. Continued

Step 1: Calculate the matrix of perturbed forward rates. The forward rates for year 1 are the same as the spot rates. For years 2 through 4, the rate is a time weighted average of the then-current perturbed spot rate and the corresponding perturbed spot rate from the prior year. For example, the perturbed forward rate f2 for year 4 is $4r_2\ (perturbed\ spot\ rate) for\ year\ 4 - 3r_2\ for\ year\ 3 = 4 \times 0.375 + 3 \times 0.035 = 0.04500.$

Perturbed forward rate for year t = t x perturbed spot rate for year t – (t-1) x perturbed spot rate for year t -1. This applies for f1, f2, f3 and f4

	perturbed forward rates			
	f1	f2	f3	f4
year 1	0.03010	0.03000	0.03000	0.03000
year 2	0.03490	0.03520	0.03500	0.03500
year 3	0.04000	0.03980	0.04030	0.04000
year 4	0.04500	0.04500	0.04470	0.04540

Step 2: Calculate the matrix of ∂P. The value for each entry is the total of the present value of the payments based on the perturbed discount factors less the total of the present value of the payments based on the given spot rates. For example, the value for the payment in year 4 for Instrument 4 is the sum-product of the payments for Instrument 4 each year and the perturbed discount factors less the sum-product of the payments for Instrument 4 each year and the discount factors based on the given spot rates.

	∂P			
	I-1	I-2	I-3	I-4
year 1	-0.0097	0.0000	-0.0004	-0.0004
year 2	0.0000	-0.0187	-0.0007	-0.0008
year 3	0.0000	0.0000	-0.0281	-0.0012
year 4	0.0000	0.0000	0.0000	-0.0360

Step 3: Calculate the transposed matrix of ∂P. This is the transposed matrix calculated in step 4 with adjustments based on the payments for each portfolio.

	∂P			
	year 1	year 2	year 3	year 4
I-1	-0.0094	0.0000	0.0000	0.0000
I-2	0.0000	-0.0182	0.0000	0.0000
I-3	-0.0004	-0.0007	-0.0272	0.0000
I-4	-0.0004	-0.0008	-0.0012	-0.0348

6. Continued

Step 4: Calculate the matrix of $\frac{\partial P}{\partial r}$. This is the transposed matrix of ∂P where each entry is divided by one basis point (0.0001).

	$\frac{\partial P}{\partial r}$			
	year 1	year 2	year 3	year 4
I-1	-94.2504	0.0000	0.0000	0.0000
I-2	0.0000	-181.6772	0.0000	0.0000
I-3	-3.7700	-7.2671	-271.8459	0.0000
I-4	-4.2413	-8.1755	-11.7626	-347.6663

Step 4b

df = perturbed forward rate – initial forward rate

	df			
	f1	f2	f3	f4
year 1	0.0001	0.0000	0.0000	0.0000
year 2	-0.0001	0.0002	0.0000	0.0000
year 3	0.0000	-0.0002	0.0003	0.0000
year 4	0.0000	0.0000	-0.0003	0.0004

Step 5: Calculate the matrix of $\frac{\partial f}{\partial r}$. This is matrix where each entry is calculated as first subtracting forward rate from the perturbed forward rate, and then dividing the difference by one basis point (0.0001).

	$\frac{\partial f}{\partial r}$			
	f1	f2	f3	f4
year 1	1.0000	0.0000	0.0000	0.0000
year 2	-1.0000	2.0000	0.0000	0.0000
year 3	0.0000	-2.0000	3.0000	0.0000
year 4	0.0000	0.0000	-3.0000	4.0000

6. Continued

Step 6: Calculate the inverse of the matrix of $\frac{\partial f}{\partial r}$.

	$\left(\frac{\partial f}{\partial r}\right)^{-1}$			
	f1	f2	f3	f4
year 1	1.0000	0.0000	0.0000	0.0000
year 2	0.5000	0.5000	0.5000	0.5000
year 3	0.3333	0.3333	0.3333	0.3333
year 4	0.2500	0.2500	0.2500	0.2500

Step 7: Calculate the matrix product of $\frac{\partial P}{\partial r}$ and $\left(\frac{\partial f}{\partial r}\right)^{-1}$

	dP/df=dP/dr*(Inverse(df/dr))			
	f1	f2	f3	f4
I-1	-94.2504	0.0000	0.0000	0.0000
I-2	-90.8386	-90.8386	0.0000	0.0000
I-3	-98.0189	-94.2488	-90.6153	0.0000
I-4	-99.1664	-94.9252	-90.8374	-86.9166

(c) Explain how the continuously compounded yield should behave (i.e., increasing or decreasing) in the following time steps if the continuously compounded forward rate is above the continuously compounded yield at the previous time steps and vice versa.

Commentary on Question:
Most candidates received credit for writing down the correct conclusion. Few candidates received credit for the derivation, or portions thereof, of the expression for the continuously compounded yield as a function of the continuously compounded forward rates.

$$Z(0,T_1) = e^{-f(0,0,T_1)\triangle}$$
$$Z(0,T_2) = Z(0,T_1) * e^{-f(0,T_1,T_2)\triangle}$$
$$\vdots = \vdots$$
$$Z(0,T_i) = Z(0,T_{i-1}) * e^{-f(0,T_{i-1},T_i)\triangle}$$
$$\vdots = \vdots$$
$$Z(0,T_n) = Z(0,T_{n-1}) * e^{-f(0,T_{n-1},T_n)\triangle}$$

6. Continued

By sequentially substituting the discount factor at previous steps, we end up the following equation:

$$Z(0, T_n) = e^{-\triangle\left(f(0,0,T_1),+f(0,T_1,T_2)+f(0,T_2,T_3)+\cdots+f(0,T_{n-1},T_n)\right)}$$

Then, the equation above is equivalent to the discount factor expressed in the continuously compounded yield with maturity of T_n:

$$Z(0, T_n) = e^{-r(0,T_n)T_n}$$

Finally, we can see that the continuously compounded yield is equivalent to the average of continuously compounded forward rates up to T_n:

$$r(0, T_n) = \frac{1}{T_n}\sum_{i=1}^{n} f(0, T_{i-1}, T_i) \triangle$$

Considering the relationship shown above, if the continuously compounded forward rates are above the continuously compounded yield at previous time steps, in the next time steps, the continuously compounded yield must be increasing, vice versa.

7. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3c) Calibrate a model to observed prices of traded securities.

(3h) Understand the application of Monte Carlo simulation to risk neutral pricing of interest rate securities.

(3j) Understand and apply the Heath-Jarrow-Morton approach including the LIBOR Market Model.

(3l) Demonstrate an understanding of the issues and approaches to building models that admit negative interest rates.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Chapter 21

QFIQ-116-17: Low Yield Curves and Absolute/Normal Volatilities

Commentary on Question:

This question is testing the candidates' understanding of the T-forward risk neutral measure and the LMM model. It also tests the candidates' knowledge on how to calibrate the volatility parameter and subsequently using the calibrated model to price fixed income derivatives.

Solution:

(a) Show that $f(t, T_1, T_2) = E^*[r(T_1, T_2)]$, where $r(T_1, T_2)$ is the spot rate from T_1 to T_2.

Commentary on Question:

Some candidates were able to proceed with the question through either the forward rate agreement setup or the martingale property of forward rate under the T-forward risk neutral measure, but many candidates were not on the right track as they did not differentiate that $r(T_1, T_2)$ is a random variable at time t, but $f(t, T_1, T_2)$ is a known value.

7. Continued

Consider a forward contract whose forward price is the forward rate $f(t, T_1, T_2)$ and at the maturity T_1 , the value is g_{T_2}.
Since the forward rate will converge to the spot rate at T_1, and forward price will converge to the value at T_1,

$$f(T_1, T_1, T_2) = g_{T_2} = r(T_1, T_2)$$

The forward price should make the contract value equal to 0 at time t, and with the given equation,

$$V_t = Z(t, T_1)E^*\left[f(t, T_1, T_2) - g_{T_2}\right] = 0$$

Thus

$$E^*\left[f(t, T_1, T_2) - g_{T_2}\right] = 0$$
$$f(t, T_1, T_2) = E^*[f(T_1, T_1, T_2)] = E^*\left[g_{T_2}\right] = E^*[r(T_1, T_2)]$$

Alternatively, the candidate can leverage the fact that forward rate is martingale under the T-forward risk-neutral measure, thus

$$f(t, T_1, T_2) = E^*[f(T_1, T_1, T_2)] = E^*\left[g_{T_2}\right] = E^*[r(T_1, T_2)]$$

(b) Determine the distribution of log($r(T_1, T_2)$) including the defining parameters.

Commentary on Question:
Candidates did relatively well on this part. Many candidates were able to derive the dynamics of log(r(T_1, T_2)) using Ito's lemma and make progress with deriving the distribution and the parameters.

Under the T_2-forward risk neutral measure, thus $m = 0$ and

$$\frac{df(t, T_1, T_2)}{f(t, T_1, T_2)} = \sigma dW_t$$

Since $r(T_1, T_2) = f(T_1, T_1, T_2)$, and based on the diffusion process above, it should have log-normal distribution.

To determine the paramters of the log-normal distribution, let $X(T_1, T_1, T_2) = \log\left(f(T_1, T_1, T_2)\right)$, and by Ito's lemma,

$$dX = \left(\frac{\partial X}{\partial t} + 0 \times \frac{\partial X}{\partial f} + \frac{\sigma^2 f^2}{2}\frac{\partial^2 X}{\partial f^2}\right)dt + \sigma f \frac{\partial X}{\partial f} dW_t = \frac{\sigma^2 f^2}{2}\left(-\frac{1}{f^2}\right)dt + \sigma f \frac{1}{f} dW_t$$
$$= -\frac{\sigma^2}{2}dt + \sigma dW_t$$

Thus $X(T_1, T_1, T_2)$ has a normal distribution with

$$mean = X(t, T_1, T_2) + \int_{s=t}^{T_1} -\frac{\sigma^2}{2}ds = \log\left(f(t, T_1, T_2)\right) - \frac{\sigma^2}{2}(T_1 - t)$$

$$variance = \sigma^2(T_1 - t)$$

7. Continued

(c) Calculate the forward volatility, σ_f^{i+1}, implied by the given caplet prices, where $i+1$ refers to the caplet maturing at T_{i+1}.

Commentary on Question:
Only some candidates were able to plug into Black's formula the appropriate values from those given in the question and solve for the answers. Correct equations with the right values plugged in but without the correct answers earned partial marks. A common mistake is that some candidates use the wrong term to maturity for the caplet price when plugging in the Black's formula.

The implied volatilities can be calculated by back solving for the volatility parameters in the Black's formula. E.g. for the first caplet

$$0.648 = \frac{1}{(1+1.0\%)^{0.25}(1+1.1\%)^{0.25}}\ (1.1\% \times N(d_1) - 1.1\% \times N(d_2))$$

Where

$d_1 = \frac{1}{\sigma\sqrt{0.5}} log\left(\frac{1.1\%}{1.1\%}\right) + \frac{1}{2}\sigma\sqrt{0.5}$ and $d_2 = d_1 - \sigma\sqrt{0.5}$

σ is goal seeked in excel and $\sigma = 21\%$

Following a similar calculation, $\sigma = 24\%$ and $\sigma = 30\%$ for the second and third caplet respectively.

(d)

(i) Simulate one outcome for each of the two forward rates in a single simulation step.

(ii) Describe how the call option on the zero-coupon bond can be priced using Monte Carlo simulations.

Commentary on Question:
Part (i) had very low attempt rate, and only few candidates were given partial marks for writing down the correct simulation formula.
For part (ii), many candidates only gave very general answers for Monte Carlo simulations without applying it under the context of the call option on the zero-coupon bond (e.g., which forward rates and what volatility parameter should be used in the simulation), which is asked by the question. These answers did not receive points, but some candidates received partial points for demonstrating the application.

7. Continued

Part (i)
With the assumption that the volatility of forward rates depends only on time to maturity, the forward rate volatility can be extracted from the caplet implied volatilties calculated above:

	$t < 0.25$	$0.25 < t < 0.5$	$0.5 < t < 0.75$
$f(t, 0.5, 0.75)$	26.6%	21%	
$f(t, 0.75, 1)$	39.3%	26.6%	21%

$$26.6\% = \sqrt{\frac{1}{0.25}(24\%^2 \times 0.5 - 21\%^2 \times 0.25)}$$

$$39.3\% = \sqrt{\frac{1}{0.25}(24\%^2 \times 0.5 - (21\%^2 + 26.6\%^2) \times 0.25)}$$

Since the call option matures at T = 0.5, we want to simulate the rates $r(0.5, 0.75)$ and $f(0.5, 0.75, 1)$ under the $T_{0.5}$-forward measure, so we can calculate the simulated bond price.
$f(t, 0.5, 0.75)$ and $f(t, 0.75, 1)$ are not martingale under $T_{0.5}$-forward measure, and their dynamics are given by

$$\frac{df(t, 0.5, 0.75)}{f(t, 0.5, 0.75)} = \frac{0.25f(t, 0.5, 0.75)\sigma_{0.75}^2}{1 + 0.25f(t, 0.5, 0.75)}dt + \sigma_{0.75}dW_t$$

where

$$\sigma_{0.75} = \begin{cases} 26.6\%, & t \leq 0.25 \\ 21\%, & 0.25 < t \leq 0.5 \end{cases}$$

And

$$\frac{df(t, 0.75, 1)}{f(t, 0.75, 1)} = \left(\frac{0.25f(t, 0.5, 0.75)\sigma_{0.75}\sigma_1}{1 + 0.25f(t, 0.5, 0.75)} + \frac{0.25f(t, 0.75, 1)\sigma_1^2}{1 + 0.25f(t, 0.75, 1)}\right)dt + \sigma_1 dW_t$$

where

$$\sigma_1 = \begin{cases} 39.3\%, & t \leq 0.25 \\ 26.6\%, & 0.25 < t \leq 0.5 \end{cases}$$

Leveraging results from part b), it can be seen that $f(0.25, 0.5, 0.75)$ can be simulated as follow:

$$f(0.25, 0.5, 0.75) = f(0, 0.5, 0.75)e^{\left(\frac{0.25f(0,0.5,0.75)0.266^2}{1+0.25f(0,0.5,0.75)} - 0.5\times0.266^2\right)\times0.25+0.266\times\sqrt{0.25}\times z_1}$$
$$= 1.2054\%$$

$$f(0.25, 0.75, 1) = f(0, 0.75, 1)e^{(m-0.5\times0.393^2)\times0.25+0.393\times\sqrt{0.25}\times z_2} = 1.2262\%$$

$$m = \frac{0.25f(0, 0.5, 0.75) \times 0.266 \times 0.393}{1 + 0.25f(0, 0.5, 0.75)} + \frac{0.25f(0, 0.75, 1)0.393^2}{1 + 0.25f(0, 0.75, 1)}$$

Part (ii)
Similarly to the simulation of $f(0.25, 0.5, 0.75)$ and $f(0.25, 0.75, 1)$ above, $f(0.5, 0.5, 0.75)$ and $f(0.5, 0.75, 1)$ can be further simulated with the appropriate volatility parameters (21% and 26.6% respectively).

7. Continued

The simulated rates can then be used to calculate price of the underlying zero-coupon bond of the option and thus the payoff at maturity:

$$Max\left(\frac{1000}{f(0.5,0.5,0.75)f(0.5,0.75,1)} - K, 0\right)$$

Repeat the simulations a large number of time (e.g., 10,000), price of the option is approximated by the average of payoffs under all simulations multiplied by the zero-coupon bond price $\frac{1}{r(0,0.5)}$.

(e) Describe two limitations of calibrating interest rate volatility using Black's formula and two potential alternatives to the LMM to address them.

Commentary on Question:
This part is done relatively well. Many candidates were able to identify the limitations and suggest a corresponding alternative.

1. When interest rates have dropped post 2008, the Black's volatilities have been observed to be very higher (30%-70% between 2009 to mid-2012, increasing to 80%-170% from mid-2012 to mid-2014, and finally greater than 200% after mid-2014). The absolute/normal volatilties have been observed to be more stable, leading to the view that interest rate distibutions may be more normal than log-normal. The diffusion process of forward rate can be modified to be:
$$df(t,T_1,T_2) = \sigma dW_t$$

2. LMM model is a log-normal model and does not allow negative rates. However negative rates have been observed recently and are becoming a possibility.
A displaced log-normal version of the Black's formula can be used to allow for negative rates in the LMM model:
$$\frac{d\widetilde{f(t,T_1,T_2)}}{\widetilde{f(t,T_1,T_2)}} = \sigma dW_t$$
where
$$\widetilde{f(t,T_1,T_2)} = f(t,T_1,T_2) + \delta$$
$-\delta$ would be the negative floor of the forward rate.

8. **Learning Objectives:**

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(3b) Understand and apply various one-factor interest rate models.

(3f) Apply the models to price common interest sensitive instruments including callable bonds, bond options, caps, floors, and swaptions.

(3i) Understand the implications of replacing LIBOR with alternatives reference rates.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Piertro, 2010

QFIQ-131-21: LIBOR replacement: Beyond LIBOR: A Primer on the New Reference Rates, BIS, March 2019

Commentary on Question:

This question was intended to test Candidates' understanding of the interest rate models and application. Correct formulas and calculations are required to earn full points. Partial credits were awarded if candidates provided correct formulas with wrong answers. In general, Candidates did not perform well on this question.

Solution:

(a) Calculate the price of a 3-month European call option on a zero-coupon bond that matures in 10 years with a face value of $100 and a strike price of $55.

Commentary on Question:

Many candidates used the wrong formula to calculate $Z(0,t)$. Some candidates calculated $B(T_O;T_B)$ and $S_Z(T_O;T_B)$ correctly, and partial points were awarded.

$$c = 100 \cdot Z(0,10)N(d_1) - K \cdot Z(0,0.25)N\big(d_1 - S_Z(T_O;T_B)\big)$$

where

$$Z(0,t) = e^{-\int_0^t f(0,\tau)d\tau} = e^{-0.05t-\frac{0.02\cdot\left(1-e^{-0.2t}\right)}{0.2}}$$
$$Z(0,0.25) = 0.98277$$
$$Z(0,10) = 0.55629$$
$$K = 55$$
$$B(T_O;T_B) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(T_B-T_O)}\right) = 3.65048$$

8. **Continued**

$$S_Z(T_O;T_B) = B(T_O;T_B) \cdot \frac{\sigma}{\sqrt{2\gamma^*}} \cdot \sqrt{1 - e^{(-2\gamma^* T_O)}} = 0.03539$$

$$d_1 = \frac{1}{S_Z(T_O;T_B)} \log\left(\frac{Z(r_0,0;T_B)}{K \cdot Z(r_0,0;T_O)}\right) + \frac{S_Z(T_O;T_B)}{2} = 0.82993$$

$$d_2 = d_1 - S_Z(T_O;T_B) = 0.79454$$

$$N(d_1) = 0.79671, N(d_2) = 0.78656$$

$$\therefore c = 1.80472$$

(b) Derive

(i) the stochastic differential equation for the forward rate $f(t,T)$ from the Hull-White Model.

(ii) the volatility of the forward rate.

Commentary on Question:
Candidates did not perform well on part (b). Most candidates failed to derive the stochastic differential equation, while some correctly derived the forward rate volatility. Most candidates were unable to apply Itô's lemma in the solution.

An affine structure is given by:

$$Z(t,T) = e^{A(t,T) - B(t,T)r(t)}$$

and

$$B(t,T) = \frac{1}{\gamma^*}\left(1 - e^{-\gamma^*(T-t)}\right)$$

Also,

$$f(t,T) = -\frac{\partial lnZ(t,T)}{\partial T} = -\frac{\partial A(t,T)}{\partial T} + \frac{\partial B(t,T)}{(\partial T)} r(t)$$
$$= -\frac{\partial A(t,T)}{\partial T} + e^{-\gamma^*(T-t)} r(t)$$

Applying Itô's lemma, we have the SDE:

$$df(t,T) = -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right] dt + \gamma^* e^{-\gamma^*(T-t)} r(t)dt + e^{-\gamma^*(T-t)} dr(t)$$
$$= -\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right] dt + \gamma^* e^{-\gamma^*(T-t)} r(t)dt$$
$$+ e^{-\gamma^*(T-t)}[(\theta_t - \gamma^* r_t)dt + \sigma dX_t]$$
$$= \left\{-\frac{\partial}{\partial t}\left[\frac{\partial A(t,T)}{\partial T}\right] + e^{-\gamma^*(T-t)}\theta_t\right\} dt + e^{-\gamma^*(T-t)}\sigma dX_t$$
$$= \alpha(t,T)dt + e^{-\gamma^*(T-t)}\sigma dX_t$$

8. Continued

So, the volatility of forward rate is

$$e^{-\gamma^*(T-t)}\sigma$$

(c)

(i) Verify the technical condition that guarantees the interest rate is always positive is satisfied.

(ii) Calculate the bond price $Z(0,10)$.

Commentary on Question:
Many candidates successfully identified the technical condition for the positive interest rates. However, most candidates did not perform well in the bond price derivation. Many candidates derived the formulas for ψ and $Z(r,0;T)$ correctly but either made mistakes on $A(t;T)$ and $B(t;T)$ calculations or didn't finish them.

Finding parameters from the question by comparing with the CIR interest rate process:

$$dr(t) = \gamma^*\left(\bar{r}^* - r(t)\right)dt + \sqrt{\alpha \cdot r(t)}dX_t$$

$$dr(t) = 0.1\left(0.03 - r(t)\right)dt + \sqrt{0.004 \cdot r(t)}dX_t$$

$$\gamma^* * \bar{r}^* = 0.1 * 0.03 = 0.003 > \frac{1}{2} * 0.004$$

$$B(t;T) = \frac{2\left(e^{\psi_1(T-t)} - 1\right)}{(\gamma^* + \psi_1)(e^{\psi_1(T-t)} - 1) + 2\psi_1}$$

$$A(t;T) = \frac{2\bar{r}\gamma^*}{\alpha}\log\left(\frac{2\psi_1 e^{(\psi_1+\gamma^*)\left(\frac{T-t}{2}\right)}}{(\gamma^* + \psi_1)\left(e^{\psi_1(T-t)} - 1\right) + 2\psi_1}\right)$$

$$\psi_1 = \sqrt{(\gamma^*)^2 + 2\alpha}$$

$$Z(r,0;T) = e^{A(0;T) - B(0;T)\times r(0)}$$

	$\boldsymbol{A(0;T)}$	$\boldsymbol{B(0;T)}$	$\boldsymbol{Z(r,0;T)}$
$T = \mathbf{10}$	-0.1081	6.0765	0.78047

(d)

(i) Describe ideal reference rate features.

(ii) Explain why LIBOR fails to meet ideal reference rate features.

(iii) Explain the attributes that the new reference rates would incorporate.

8. Continued

Commentary on Question:
Most candidates listed features and attributes partially. Part (d) performance was the highest among all parts of the question.

i. Ideal reference rate features

The ideal reference rate would have to:

1) Provide a robust and accurate representation of interest rates in core money markets that is not susceptible to manipulation
2) Offer a reference rate for financial contracts that extend beyond the money market
3) Serve as a benchmark for term lending and funding

ii. Why LIBOR fails to meet them

LIBOR fulfills the second and the third features imperfectly. However, it fails to meet the first criterion for four reasons.

1) LIBOR was constructed from a survey of a small set of banks reporting non-binding quotes rather than actual transactions. It is a design flaw.
2) Sparse activity in interbank deposit markets stood, and still stands, in the way of a viable transaction-based benchmark based on interbank rates.
3) The increased dispersion of individual bank credit risk since 2007 has undermined the adequacy of benchmark such as LIBOR that aim to capture common bank risk, even for users seeking a credit risk exposure.
4) Due to regulatory and market efforts to reduce counterparty credit risk in interbank exposures, banks have also tilted their funding mix towards less risky sources of wholesale funding.

1) Shorter tenor: essentially by moving to O/N markets, where volumes are larger than for longer dated tenors such as three months
2) Moving beyond interbank markets: to add bank borrowing from a range of non-bank wholesale counterparties
3) In some jurisdictions, drawing on secured rather than unsecured transactions. The secured transactions could also include banks' repurchase agreements with non-bank wholesale counterparties

9. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

3. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(3a) Understand and apply the concepts of risk-neutral measure, forward measure, normalization, and the market price of risk, in the pricing of interest rate derivatives.

(3f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3j) Understand and apply the Heath-Jarrow-Morton approach including the LIBOR Market Model.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010, Chapter 20 and Chapter 21

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014, Chapter 9 and Chapter 10

Commentary on Question:

Commentary is listed underneath each question component.

Solution:

(a) Describe

(i) Forward Risk Neutral Pricing methodology.

(ii) The LIBOR market model (LMM) and its inputs and methodologies for pricing caps, floors and complicated securities.

9. Continued

Commentary on Question:
Candidates performed as expected on this question. Many candidates only answered part of the questions in part (ii).

(i) In the Forward Risk Neutral Pricing methodology, we express the value of securities in terms of the price of another conveniently chosen security (change of numeraire), usually a zero-coupon bond with the same maturity. Under the T-forward risk neutral dynamics, the forward price for delivery at T is a martingale.

(ii) The LIBOR market model specifies the dynamics of LIBOR-based forward rates. Each forward rate with maturity T is a martingale and follows a lognormal diffusion process under the T-forward risk neutral dynamics.

LMM provides a no arbitrage derivation of the Black formula to price caps and floors. The inputs are caplet implied volatilities and the current term structure of interest.

One can use Monte Carlo simulations to price complicated securities.

(b) Compute the value of the above caplet at time 0 under the LMM.

Commentary on Question:
Candidates performed as expected on this question. Note that in the caplet pricing formula provided in the formula sheet, $f_n(0,\tau,T)$ is n-times compounded, not continuously compounded.

$\text{Caplet}(0;T_{i+1}) = N\Delta\, Z(0,T_{i+1})[f_n(0,T_i,T_{i+1})\, N(d_1) - r_K N(d_2)]$, where
$N = 100{,}000, T_i = 1.5, T_{i+1} = 2, \Delta = 2 - 1.5 = 0.5, n = 1/\Delta = 2, r_K = 0.03.$

To calculate $f_2(0, 1.5, 2)$, we must first calculate the price of zero-coupon bonds.

Note: Because no compounding frequency is specified for the LIBOR spot rate in this question, points were awarded for all compounding frequencies in the calculation of $Z(0{,}1.5)$ and $Z(0{,}2)$. In this solution, we present a version that assumes continuous compounding.

$$Z(\tau,T) = \exp\big(-r(T-\tau)\big)$$
$$Z(0,1.5) = \exp(-0.02 \times 1.5) = 0.9704$$
$$Z(0,2) = \exp(-0.02 \times 2) = 0.9608$$

9. Continued

$f_n(0, T_i, T_{i+1})$ is the *n*-times compounded forward rate at time 0 for an investment at T_i maturing at T_{i+1}.

$$f_2(0, 1.5, 2) = 2 \times \frac{Z(0, 1.5) - Z(0, 2)}{Z(0, 1.5)} = 0.0201$$

Next, we calculate $N(d_1)$ and $N(d_2)$:

$$\sigma_f \sqrt{T_i} = 0.3 \times \sqrt{1.5} = 0.3674$$

$$d_1 = \frac{1}{\sigma_f \sqrt{T_i}} ln\left(\frac{f_2(0, T_i\, T_{i+1})}{r_k}\right) + \frac{1}{2}\sigma_f \sqrt{T_i} = -0.9062$$

$$d_2 = d_1 - \sigma_f \sqrt{T_i} = -1.2736$$

$$N(d_1) = 0.1824, N(d_2) = 0.1014$$

The price of the caplet is then

$$100{,}000 \times 0.5 \times 0.9608[0.0201 \times 0.1824 - 0.03 \times 0.1014] = 29.99$$

(c) Express the convexity adjustment term in terms of $f_n\ (0, \tau\ , T)$, σ_f, Δ and τ.

Commentary on Question:
Candidates performed poorly in this question. In this question, candidates are asked to express the expected spot rate $r_n(\tau, T)$ under the τ-forward risk neutral dynamics as the forward rate $f_n(0, \tau\ , T)$ plus a convexity adjustment. Many candidates wrote down the formula for expressing continuously compounded forward rate in terms of the futures rate, which is not correct.

$$E_f^{*\tau}[r_n(\tau, T)] \approx f_n\ (0, \tau\ , T) + \frac{f_n\ (0, \tau\ , T)^2\ \Delta}{1 + f_n\ (0, \tau\ , T)\Delta}\ \sigma_f^2\ \tau$$

(d) Calculate the value of the LIBOR-in-arrears swap by using part (c).

Commentary on Question:
Most candidates did not attempt this question.

Because it is an in-arrears swap, convexity adjustment must be made. The convexity adjustment changes the forward rate assumed at time T_i from 0.03 to the following

$$0.03 + \frac{0.03^2 \times\ 1 \times (0.2)^2 \times T_i}{1 + 0.03 \times 1} = 0.03 + 0.00003495\ T_i$$

where $\Delta = 1, T_i = 1, 2, 3, 4$.

9. Continued

	Year 1	Year 2	Year 3	Year 4
Fixed	0.03	0.03	0.03	0.03
Floating	0.03003495	0.03006990	0.03010485	0.03013980
Fixed – Floating	−0.00003495	−0.00006990	−0.00010485	−0.00013981
Discount Factor	$\frac{1}{1.03}$	$\frac{1}{1.03^2}$	$\frac{1}{1.03^3}$	$\frac{1}{1.03^4}$

The value of the LIBOR-in-arrears swap is then

$$-\$100 \text{ million} \times \left(\frac{0.00003495}{1.03} + \frac{0.00006990}{1.03^2} + \frac{0.00010485}{1.03^3} + \frac{0.00013981}{1.03^4}\right)$$
$$= \$ - 31{,}999.59$$

(e)

(i) Show by using Ito's lemma that

$$dS_{0,2}(t) = \sum_{i=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_i} \sigma_i(t) F_i(t) dW_i(t)$$

Hint: Consider the diffusion term of $dS_{0,2}(t)$ and use the following differential equations:

$$dF_i(t) = \mu_i^A(t) F_i(t) dt + \sigma_i(t) F_i(t) dW_i(t); \; i = 1, 2$$

(ii) Show by using part (i) and $dS_{0,2}(t) = \sigma_{0,2}(t) S_{0,2}(t) dW(t)$ that

$$\sigma_{0,2}(t)^2 = \frac{\sum_{i,j=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_i} \frac{\partial S_{0,2}(t)}{\partial F_j} \sigma_i(t) \sigma_j(t) F_i(t) F_j(t) \rho_{i,j}}{S_{0,2}(t)^2}$$

where $\rho_{1,2} = \rho_{2,1} = \rho$ and $\rho_{1,1} = \rho_{2,2} = 1$

(iii) Assess, by considering the distribution of $\sigma_{0,2}(t)$ in part (ii) whether the LMM and the SMM are compatible.

Commentary on Question:
Most candidates did not attempt this question.

9. Continued

(i) By Ito's Lemma

$$dS_{0,2}(t) = \sum_{i=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_i} dF_i(t) + (\ldots)dt$$

Substiute the below SDE into $dS_{0,2}(t)$, we obtain

$$dF_i(t) = \mu_i^A(t)F_i(t)dt + \sigma_i(t)F_i(t)dW_i^A(t)$$

All the terms containing dt disappear because $S_{0,2}(t)$ is a martingale and the drift terms must be 0. $W_i^A(t)$ are Brownian motions under the swap measure P_A. Therefore,

$$dS_{0,2}(t) = \sum_{i=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_i} \sigma_i(t)F_i(t)dW_i^A(t)$$

(ii) From $dS_{0,2}(t) = \sigma_{0,2}(t)S_{0,2}(t)dW(t)$, we obtain

$$\sigma_{0,2}(t)dW(t) = \frac{dS_{0,2}(t)}{S_{0,2}(t)}$$

Square both sides, we have

$$\sigma_{0,2}(t)^2 dt = \frac{dS_{0,2}(t)}{S_{0,2}(t)} \frac{dS_{0,2}(t)}{S_{0,2}(t)}$$

$$= \frac{\sum_{i=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_i} \sigma_i(t)F_i(t)dW_i^A(t)}{S_{0,2}(t)} \frac{\sum_{j=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_j} \sigma_i(t)F_j(t)dW_j^A(t)}{S_{0,2}(t)}$$

$$= \frac{\sum_{i,j=1}^{2} \frac{\partial S_{0,2}(t)}{\partial F_i} \frac{\partial S_{0,2}(t)}{\partial F_j} \sigma_i(t)\sigma_j(t)F_i(t)F_j(t)\rho_{i,j}}{S_{0,2}(t)^2} dt$$

Since $dW_i^A(t)dW_j^A(t) = \rho_{i,j}dt$.

(iii) In the LMM, $\sigma_{0,2}(t)$ is clearly not a deterministic function, which shows that the swap rate $S_{0,2}(t)$does not follow a log-normal process under P_A. Therefore, LMM is not compatible with the SMM.

10. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(4e) Analyze the Greeks of common option strategies.

(4g) Describe and explain some approaches for relaxing the assumptions used in the Black-Scholes-Merton formula.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-114-17: Chapter 2, pp. 162-173 and 223-225 of Frequently Asked Questions in Quantitative Finance, Wilmott, Paul, 2nd Edition, 2009

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Determine the notional amount of the variance swap necessary to hedge the existing volatility swap. Assume you can sell variance swaps on the ABC Index at the strike of 17%.

Commentary on Question:

This is a straightforward retrieval question that tests candidates' basic understanding of hedging. Candidates did well on the question. Most candidates used the correct formula to calculate the notional of the variance swap.

The notional of the variance swap is

$$N_{var} = \frac{1}{2\sigma_K} N_{vol}$$

$$= \frac{1}{2(.17)}(1{,}000{,}000)$$

= 2,941,176

10. Continued

(b) Determine the payoff on the hedged position if the realized volatility is 24%.

Commentary on Question:
This question tests candidates understanding of how realized volatility impacts hedging position. Candidate did fairly well on this question. Full points were awarded to candidates who used the correct notionals in the formula.

The payoff of the hedged position is

$$\pi = N_{vol}(\sigma_R - \sigma_K) - N_{var}(\sigma_R^2 - \sigma_K^2)$$

$$= 1{,}000{,}000(.24 - .17) - 2{,}941{,}176(.24^2 - .17^2)$$

$$= -14{,}411.75$$

(c) Show the present value of P&L at time t_0 is given by

$$PV[P\&L(i,h)] = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{\{-r(t-t_0)\}}\ \Gamma_h S^2(\sigma_R^2 - \sigma_h^2)dt.$$

Commentary on Question:
To get full points for this question, candidates need to (1) clearly demonstrate how Ito's Lemma is applied in hedging a long volatility swap; (2) identify what needs to be satisfied for BSM with hedged volatility; (3) correctly take present value of P&L and integrate over the life of the option. This question is challenging. Very few candidates attempted this question.

$$\begin{aligned}
\mathrm{dP\&L} &= \mathrm{dV_i} - \Delta_\mathrm{h}\mathrm{dS} - \Delta_h \mathrm{SDdt} + [(\Delta_\mathrm{h}\mathrm{S} - \mathrm{V_h}) + (\mathrm{V_h} - \mathrm{V_i})]\mathrm{rdt} \\
&= (\mathrm{dV_h} - \mathrm{dV_h}) + \mathrm{dV_i} - \Delta_\mathrm{h}\mathrm{dS} - \Delta_\mathrm{h}\mathrm{SDdt} + \Delta_\mathrm{h}\mathrm{Srdt} - \mathrm{V_h}\mathrm{rdt} + (\mathrm{V_h} \\
&\quad - \mathrm{V_i})\mathrm{rdt} \\
&= \mathrm{dV_h} - \Delta_\mathrm{h}\mathrm{dS} + \Delta_\mathrm{h}(\mathrm{r} - \mathrm{D})\mathrm{Sdt} - \mathrm{V_h}\mathrm{rdt} + (\mathrm{dV}_i - \mathrm{dV_h}) + (\mathrm{V_h} - \mathrm{V_i})\mathrm{rdt}
\end{aligned}$$

By Ito's Lemma

$$(\mathrm{dV_h} - \Delta_h\mathrm{dS}) = (\Theta_\mathrm{h} + \frac{1}{2}\Gamma_\mathrm{h}\mathrm{S}^2\sigma_\mathrm{r}^2)\mathrm{dt}$$

Substituting this into P&L gives

$$\begin{aligned}
\mathrm{d[P\&L]} &= (\Theta_\mathrm{h} + \frac{1}{2}\Gamma_\mathrm{h}\mathrm{S}^2\sigma_\mathrm{r}^2)\mathrm{dt} + \Delta_\mathrm{h}(\mathrm{r} - \mathrm{D})\mathrm{Sdt} - \mathrm{V_h}\mathrm{rdt} + (\mathrm{dV_i} - \mathrm{dV_h}) + (\mathrm{V}_h - \mathrm{V_i})\mathrm{rdt} \\
&= (\Theta_\mathrm{h} + \Delta_\mathrm{h}(\mathrm{r} - \mathrm{D})\mathrm{S} + \frac{1}{2}\Gamma_\mathrm{h}\mathrm{S}^2\sigma_\mathrm{r}^2 - \mathrm{V_h}\mathrm{r})\mathrm{dt} + (\mathrm{dV_i} - \mathrm{dV_h}) + (\mathrm{V_h} - \mathrm{V_i})\mathrm{rdt}
\end{aligned}$$

10. Continued

The BSM with hedged volatility satisfies

$$\Theta_h + \Delta_h(r - D)S + \frac{1}{2}\Gamma_h S^2 \sigma_h^2 - rV_h = 0$$

Thus

$$\begin{aligned} d[P\&L] &= \frac{1}{2}\Gamma_h S^2(\sigma_r^2 - \sigma_h^2)dt + (dV_i - dV_h) + (V_h - V_i)rdt \\ &= \frac{1}{2}\Gamma_h S^2(\sigma_r^2 - \sigma_h^2)dt + e^{rt}d[e^{-rt}(V_h - V_i)] \end{aligned}$$

Take present values to obtain

$$dPV[P\&L] = e^{-r(t-t_0)}\frac{1}{2}\Gamma_h S^2(\sigma_r^2 - \sigma_h^2)dt + e^{rt_0}d[e^{-rt}(V_h - V_i)]$$

and integrate over the life of the option to get

$$PV[P\&L(i,h)] = V_h - V_i + \frac{1}{2}\int_{t_0}^{T} e^{-r(t-t_0)}\,\Gamma_h S^2(\sigma_r^2 - \sigma_h^2)dt$$

(d) Calculate the standard deviation of the hedging error when rebalancing daily.

Commentary on Question:
This question tests candidates' understanding of rebalancing technique and calculation. Most candidates used the correct formula to calculate the vega of the option as well the standard deviation of the hedging error but failed to identify that daily rebalancing for 3 months results in a total number of 63 reblancings.

Calculate the vega of the option

$$\frac{\partial C}{\partial \sigma} = \frac{S\sqrt{\tau}}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}$$

$$= 797.09$$

The standard deviation of the hedging error is approximately

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}}\frac{\sigma}{\sqrt{n}}\frac{\partial C}{\partial \sigma} = \frac{141.28}{\sqrt{n}}$$

We are interested in daily rebalancing for 3 months, or 63 rebalancings. Hence $\sigma_{HE} = 17.80$

10. Continued

(e) Calculate the price of the call option assuming transaction costs of 1 basis point and daily rebalancing.

Commentary on Question:
This question tests candidates' comprehension of transaction costs and its impact on hedging. Most candidates who attempted this question made the similar mistake as part d, where the total number of rebalancings should be 63.

We have

$$\tilde{\sigma} \approx \sigma - k\sqrt{\frac{2}{\pi dt}} = 0.20 - 0.0001\sqrt{\frac{2}{\pi dt}} = 0.20 - 0.0001\sqrt{\frac{2(63)}{\pi}} = 0.1937$$

Thus

$$C(\widetilde{S, K, v}) = 4000 \times N(0.04984) - 4000 \times N(-0.04984) = 159.01$$

11. **Learning Objectives:**

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4e) Analyze the Greeks of common option strategies.

(4i) Define and explain the concept of volatility smile and some arguments for its existence.

Sources:

The Volatility Smile, Derman Miller Park, Ch 6, Ch 7, Ch 9, Ch 10, page 169

QFIQ-120-19 Chapter 6 and 7 of Pricing and Hedging, page 133

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a)

(i) Identify an arbitrage opportunity that is implied by your analyst's curve.

(ii) Show how you can take advantage of the arbitrage opportunity implied by your analyst's curve.

Commentary on Question:

Majority of candidates correctly calculated the put option prices. Some candidates lost points because of not explaining the profit resulting from the strategy, or because of not explaining why the arbitrage opportunity exists or not stating clearly that it should be traded until the market corrects itself. Partial credits are granted.

11. Continued

60 Strike Put = 6.65
70 Strike Put = 6.52
80 Strike Put = 7.26
90 Strike Put = 8.97
100 Strike Put = 11.92

Based on this curve, an arbitrage opportunity exists as the put option price - should increase as the strike increases which is not the case from the 60-strike put to the 70-strike put. Therefore, if the 70-strike put is reasonable then the 60-strike put is overpriced, otherwise the 70-strike put is underpriced.

To take advantage of this opportunity, one can trade a put spread by selling a 60-strike put and buying 70-strike put to lock in 6.65-6.52 =0.13 in profit for each spread traded. This should be traded as much as possible until the market price adjusts to eliminate this arbitrage opportunity.

There are no other arbitrage opportunities as the remainder of the put prices increase with the strike.

(b) Calculate the initial number of shares you need to trade. Clarify your trade as "buy" or "sell".

Commentary on Question:
Majority of candidates did not have the correct formula for the Put Delta and instead utilized the $\Delta Put_{BSM}+$ *as the Put Delta.*

Put Delta = $\Delta Put_{BSM} + \frac{\partial Put_{BSM}}{\partial \sigma}\frac{\partial \sigma}{\partial S}$
where
$\Delta Put_{BSM} = -N(-d1)$
= -0.4404

$\frac{\partial Put_{BSM}}{\partial \sigma} = Vega_{BSM}$
$= 0.01 * S\sqrt{T-t}N'(d1)$
$= 0.01 * 100 * \sqrt{1}\frac{1}{\sqrt{2\pi}} * e^{-\frac{0.15^2}{2}}$
= 0.3945

$\frac{\partial \sigma}{\partial S} = \frac{0.3*1.6}{K^{1.6}}S^{1.6-1}$
$=\frac{0.3*1.6}{100^{1.6}}100^{1.6-1}$
=0.0048

11. Continued

Put Delta = 1000*(-0.4404 + 0.3945*0.0048)
= -438.49

Therefore, the trade is to sell 438.49 shares (rounding is acceptable).

12. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4e) Analyze the Greeks of common option strategies.

(4i) Define and explain the concept of volatility smile and some arguments for its existence.

(4j) Compare and contrast "floating" and "sticky" smiles.

Sources:

QFIQ-120-19: Pricing and Hedging Financial Derivatives, Marroni, L. & Perdomo, I

Commentary on Question:

This question tests candidate's knowledge of volatility skew and the hedging strategies in light of the skew.

Solution:

(a)

(i) Identify the common name of the above observation.

(ii) Explain one reason for this observation.

Commentary on Question:

Candidates performed as expected on this part.

(i) The observation is referred to as volatility skew, where volatilities of OTM put are higher than ATM volatilities which are higher than volatilities of OTM calls.

(ii) Lower strike options tend to have higher implied volatilities than higher strike options, which can be thought of as the additional risk premium the buyer is willing to pay for the protection from a market downturn. This reflects the typical belief that a falling stock mark is likely to be more volatile than a rising market

(b) Your colleague made the following comment:

"If the volatility is constant, there is no need to volatility hedge. But since the volatility can change by strike price, it's also important to ensure that the total Vega exposure of the portfolio is zero."

Critique the comment of your colleague.

12. Continued

Commentary on Question:
Candidates performed below expectations on this part.

The colleague is not correct about the volatility hedging strategy being all about eliminating the Vega exposure.

Making the portfolio Vega-neutral can only immunize one's position with respect to parallel shifts in the volatility skew but is not going to mitigate the risk of a change in the slope or the convexity of the volatility skew.

Vega hedging is not simply offsetting the overall Vega exposure. It needs to focus on higher-order derivatives such as dVega/dSpot or dVega/dVol.

(c)

(i) Choose one strategy from the above that you think is the most appropriate. Explain why.

(ii) Plot the Vega as a function of the underlying price.

(iii) Calculate the maximum and minimum in the plot.

Commentary on Question:
Candidates performed below expectations on this part because many did not construct the Vega plot on the Excel sheet.

(i) It is a steepening volatility skew. Therefore, a Risk Reversal strategy should be used, consisting of a long position in the 110 Call and a short position in the 90 Put option.

(ii) Given $Vega = S\sqrt{t}\,N'(d_1)$ where

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

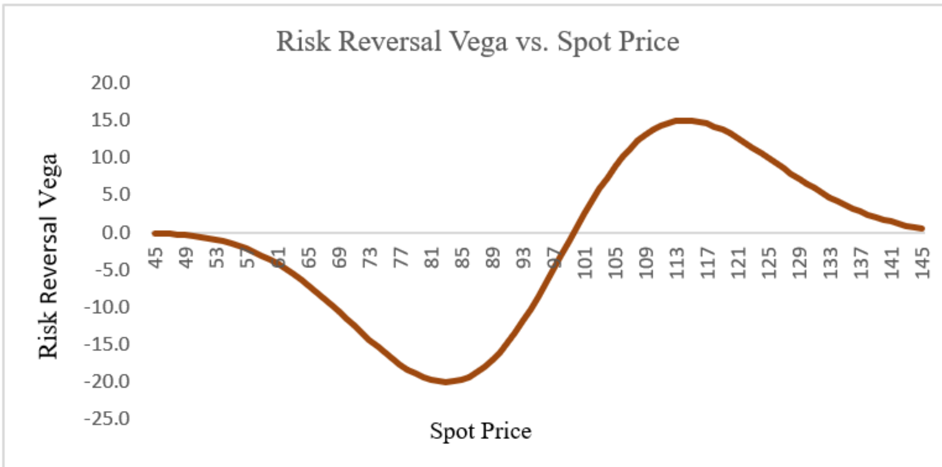
which is a symmetric function. Vega of the Call/Put option must be symmetric.

The maximum should be reached at around S > 100, the reason being that the Vega of a short 90 Put is increasing when S > 90, which contributes to the increase of the sum.

12. Continued

Similarly, the minimum is expected to be reached at around S < 90 as the decrease in the Call Vega when S < 110 which contributes to the decrease of the sum.

The plot should look as below.



a. Index spot price (S)	45	46	…	82	83	84	…	113	114	115	…	143	144
b. Vega (call 110 with σ = 18%)	0.00	0.00	…	2.22	2.74	3.36	…	29.94	29.41	28.75	…	3.56	3.17
c. Vega (put 90 with σ = 30%)	0.10	0.14	…	22.21	22.81	23.35	…	15.03	14.40	13.78	…	2.64	2.46
d. Risk Reversal Vega = b - c	-0.10	-0.14	…	-19.99	-20.07	-19.99	…	14.91	15.01	14.97	…	0.92	0.72

Note: Only partial values were shown in the above table for illustration purpose.

(iii) Based on the above table:
Maximum Risk Reversal Vega = 15.01 @ S = 114
Minimum Risk Reversal Vega = -20.07 @ S = 83

(d)

(i) Choose one strategy from the above that you think is the most appropriate. Explain why.

(ii) Describe the construction of the strategy. Specify the long/short position of the strategies involved.

(iii) Explain why the strategy is not Vega-neutral and describe, without calculation, how to construct it to be Vega-neutral

12. Continued

Commentary on Question:
Many candidates were able to identify Butterfly strategy as the appropriate strategy, but few were able to explain how to make the strategy vega-neural.

(i) It is a **change in convexity** in the volatility skew. Therefore, a Butterfly is most appropriate due to its high level of $dVega/dVol$

(ii) The Butterfly can be constructed using the long position in a strangle and a short position in a straddle. In the 90-100-110 butterfly, we take long 110 Call and a 90 Put, short a 100-strike straddle.

(iii) The Butterfly constructed this way is not Vega-neutral as the vega of the straddle is higher than the strangle. To construct a vega-neutral Butterfly, we can set the ratio of the notional of the straddle over the notional of the strangle equal to the ratio of the vega:

$$\frac{Straddle\ notional}{Strangle\ notional} = \frac{Straddle\ Vega}{Strangle\ Vega}$$

13. Learning Objectives:

4. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(4a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(4d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(4e) Analyze the Greeks of common option strategies.

(4h) Compare and contrast the various kinds of volatility, e.gl, actual, realized, implied and forward, etc.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

Commentary on Question:

Commentary listed underneath question component.

Solution:

(a) Construct a strategy to replicate the payoff of the contingent claim with only European options on Stock XYZ.

Commentary on Question:

Most candidates were able to identify the correct European call positions to replicate the contingent claim. Partial credit is granted for a correct but incomplete specification of parameters (long/short, strike, maturity) strategy.

The strategy required to replicate the payoff of the contingent claim consists of the following positions:

- A long position in a 100-strike one-year European call
- A short position in a 120-strike one-year European call

13. **Continued**

(b) Compare and contrast realized volatility and implied volatility.

Commentary on Question:
Most candidates answered this question correctly and were able to provide the clear definition for the two types of volatilities.

Implied volatility is a parameter that matches the model option price to the market price using the Black-Scholes Model equation. Implied volatility is derived from the present and expected future data.

Realized volatility is a statistic that measures the standard deviation of returns for a past period. Realized volatility is derived from the past/historical data.

(c)

(i) Calculate the Delta of this contingent claim.

(ii) Explain why the Delta is positive.

Commentary on Question:
Most candidates understood that the Delta of the contingent claim was the sum of the Deltas of the long and short European call positions from part (a). Some candidates did not calculate the correct values of the delta for the long and short European call positions.

For the overall Delta of the contingent claim, it was important to understand the relationship of the Delta of the two European call positions and how that impacts the overall Delta of the contingent claim. Most candidates only noted one or the other.

The Delta of the contingent claim is calculated as the sum of the delta of the replicating strategy in which

$$\Delta_{\text{Call}(K=100)} = \text{N}(d_1) = \text{N}\left(\frac{\ln\frac{110}{100} + \left(0 + \frac{0.3^2}{2}\right)(1)}{0.3\sqrt{1}}\right) = \text{N}(0.46770) = 0.68$$

$$\Delta_{\text{Call}(K=120)} = \text{N}(d_1) = \text{N}\left(\frac{\ln\frac{110}{120} + \left(0 + \frac{0.3^2}{2}\right)(1)}{0.3\sqrt{1}}\right) = \text{N}(-0.14004)$$
$$= 0.44432$$

13. Continued

Therefore, the Delta of the contingent claim is:

$$\Delta_{\text{Claim}} = +\Delta_{\text{Call(K=100)}} - \Delta_{\text{Call(K=120)}}$$
$$= +0.68 - 0.44432$$
$$= 0.23568 \approx \boxed{0.24}$$

The overall Delta of the contingent claim is positive because:

- A call option with a lower strike will always have a Delta that is equal to or higher than a call with a higher strike.
- Since the call option that is long has a higher Delta than the call option that is short, the resulting net Delta of the contingent claim is positive.

(d) Regarding a long position in the contingent claims, your colleague made the following comments:

- Comment 1: As the price of the underlying stock moves away from the price range within the two strike prices, we expect the Delta of the contingent claim to converge to zero.
- Comment 2: The net Gamma exposure of the contingent claim is always positive.

Assess each of your colleague's comments above.

Commentary on Question:
For Comment 1, it could be successfully approached by either describing the Delta for the two European call options and the net impact of those two Deltas or describing the Delta of the contingent claim. Most candidates took the first approach.

For Comment 2, it is important to state when Gamma for the contingent claim goes from negative to positive and vice versa. Some candidates stated the Gamma could be negative without providing additional details.

Comment 1 is correct.

This is because any further movement of the price of the underlying asset above the higher strike of 120 or below the low strike of 100 will not have any meaningful effect on the contingent claim payoff.

13. **Continued**

Comment 2 is incorrect.

The net Gamma exposure of the contingent claim will switch from positive to negative when the underlying price moves from the lower strike of 100 to the higher strike of 120. Conversely, the net Gamma exposure of the contingent claim will switch from negative to positive when the underlying price moves from the higher strike of 120 to the lower strike of 100.

(e)

(i) Calculate the profit or loss at the end of the next day from Delta hedging.

(ii) Explain why the profit or loss is not zero from Delta hedging.

Commentary on Question:
The Delta for hedging the contingent claim is the same as from part (c). Most candidates calculated the payoff of the contingent claim correctly. Some candidates did not calculate the final loss due to either not scaling the number of shares needed for delta hedging or not using the correct Delta.

Regarding why the loss is not zero from Delta hedging. Most candidates identified that the other Greeks were not hedged. Not many candidates identified that the large movement in the underlying contributes to the non-zero loss.

To Delta hedge the 100 contingent claims, the firm needs to short 24 shares. The answer of 24 shares is derived from 100 contingent claims x 0.24 (delta of the contingent claim).

The profit or loss from Delta hedging is then calculated as:

$$\begin{aligned}\text{Profit} &= 100[(33.56-20.40)-(18.14-9.28)]-24(130-110)\\ &= \boxed{-50}\end{aligned}$$

The loss is not zero from Delta hedging in that:

- There is a large movement in the price of the underlying.
- The firm has only Delta hedged and did not hedge the other Greeks.

14. Learning Objectives:

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(5a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

(5c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:
- (i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.
- (ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

Sources:

QFIQ-134-22: An Intro to Computational Risk Management of Equity-Linked Insurance (Ch 1.2-3, 4.7-8, 6.2-3)

QFIQ-135-22: Structured Product Based Variable Annuities (Sections 2 & 3)

QFIQ-128-20: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Commentary on Question:

Commentary listed underneath question component.

Solution:

Payoff of design 1 = D1 = $\max(1.035^{\wedge 2} * S_o, S_2)$
Payoff of design 2 = D2 = $\max(S_o, S_1, S_2)$

P(D2 > D1) = $P(\max(1.035^{\wedge 2} * S_o, S_2) < \max(S_o, S_1, S_2))$
$= P(\max(1.035^{\wedge 2} * S_o - S_2, 0) < \max(S_o - S_2, S_1 - S_2, 0))$

Therefore: D2 > D1 if and only if $S_1 > 1.035^{\wedge 2} * S_o$ and $S_1 > S_2$
Note: $S_2 > S_1$ and $S_2 > 1.035^{\wedge 2} * S_0$ leads to D1 = D2

Under the BS framework S_1/S_0 is independent of S_2/S_1 and both are lognormally distributed
P(D2 > D1) = $P(S_1 > S_2$ and $S_1 > 1.035^{\wedge 2} * S_0) = P(1 > S_2/S_1$ and $S_1/S_0 > 1.035^{\wedge 2})$

14. Continued

Due to independence of returns under the BS framework:
$P(1 > S_2/S_1 \text{ and } S_1/S_0 > 1.035^{\wedge 2}) = P(S_1 > S_2) * P(S_1 > 1.035^{\wedge 2} * S_0)$ (*)

$P(S_1 > S_2) = P((Ln(S_2/S_1)-r)/\sigma < -r/\sigma)$ where $((Ln(S_1/S_2)-r)/\sigma \sim N(0,1)$ under the BS framework

Therefore: $P(S_1 > S_2) = N(-0.03/.15) = .42074$

$P(S_1 > 1.071225*S_0) = P((Ln(S_1/S_0)-r)/\sigma > (\ln(1.071225)-r)/\sigma) = 1 - N(0.259) = .397939$
$P(D2 > D1) = .42074 * .397939 = 0.167429$

(b)

(i) Calculate the guaranteed benefit bases (in the table above) for the designs under consideration.

(ii) Compare the performance of the different guarantee bases during a prolonged equity down-turn.

Commentary on Question:
Candidates did well on this part.

Reset formula: $G_t = \max(G_{t-1}, S_t)$
Roll-up formula: $G_t = (1+k)*G_{t-1}$
Annual High Step up: $G_t = \max\{G_{t-1}, G_{t-1}(S_t/S_{t-1})\}$

	0	1	2	3	4
S_t	100	95	80	105	125
Reset	100	100	100	105	125
Roll-up (k = 3% annual)	100	103	106.09	109.27	112.5509
Annual High Step up	100	100	100	131.25	156.25

During a prolonged equity down-turn the reset and annual high step up would remain flat, while the roll-up would continue to increase at 3% per year.

14. **Continued**

(c)

(i) Describe two types of policyholder behavior that could impact this type of product.

(ii) Describe how GVA could design their product to mitigate the risk of these behaviors in part (c)(i).

Commentary on Question:
Candidates did well on this part. Candidates who provided other risks to the company got credit for this part as well.

(i) Surrenders: Policyholder surrender timing can both cause losses due to selective lapsing based on the moneyness of the guarantee or by causing disruption of illiquid assets in the hedge portfolio.

Withdrawal timing: withdrawals timing can cause losses based on the relative moneyness of the fund value, particularly during the early years.

(ii) Surrenders can be partially managed through the use of surrender charges.

Withdrawal timing can be partially managed by including a deferral period in the product that delays when the policyholder can begin their guaranteed withdrawals.

(d) Construct a self-financing hedge portfolio, at time t=0, for the GMWB product based on the CRO's suggestion, and the parameters (in table above) determined by the company's hedging area.

Commentary on Question:
Candidates did not do well on this part. Most candidates did not calculate the position in the 1-year zero-coupon bond (ZCB) correctly.

<u>At time t = 0</u>
Position in stock = $\Delta_t = \partial L_t / \partial S_t$ = 0.5

$\rho_t = \frac{\partial Lt}{\partial rt} = -.02$

$\rho_t^B = {}^{\partial P_{t,t+1}}/_{\partial rt} = \partial e^{-r*1} = e^{-r*1} = -e^{-0.03}$

14. Continued

Position in 1-year ZCB = $n_t = {}^{\rho_t}/_{\rho_t^B} = -.02/-e^{-.03} = .020609$

Position in Bank account = $-\Delta_t S_t - n_t P_{t,t+1}$ since the portfolio is self-funding and thus $\Pi_0 = 0$.

Position in Bank account = -0.5*100 - .020609 * $e^{-.03}$ = -50.02

(e)

(i) Calculate the net present value of the distributable earnings and the profit margin using a hurdle rate of 8%.

(ii) Critique the profit-testing approach used, in part (e)(i), to evaluate the profitability of the VA product.

Commentary on Question:
Candidates did well on part (i). Full credit was given to candidates who divide the NPV by a number (assumed to be the premium) to obtain the profit margin (PM). On part (ii), most candidates did not state that investment risks cannot be diversified through large pool of policies.

(i) NPV = ∑distributable earnings(t) /(1+hurdle rate)^t
=-1200/1.08 + 200/1.08^2+…+700/1.08^7
= 719.35
PM = NPV / premium

(ii) This profit testing is using deterministic assumptions (best estimates) including mortality rates, lapse rates, interest rates and so on. It normally works well for pricing traditional products with mostly mortality risk that can be diversified through a large pool of policyholders according to the law of large numbers. It allows an insurer to obtain ballpark estimates of its profits over long run.

However, investment risk associated with equity-linked insurance products is not diversifiable. It is impossible to use a single scenario to capture the uncertainty with equity returns.

We need to replace the deterministic scenario by stochastic scenarios to calculate the profit measures from all scenarios and form an empirical distribution of the insurer's profit measure.

14. Continued

(f)

(i) Contrast the similarity and difference between actuarial profit-testing and no-arbitrage pricing practices.

(ii) Explain the reasons that no-arbitrage pricing is not directly used in the insurance industry.

Commentary on Question:
Candidates did not do well on this part. For part (ii) most candidates listed only that no arbitrage pricing does not allow for expenses or profits.

(i) No-arbitrage pricing and actuarial pricing practices are very similar in that both aimed to find the fair risk charge such that the net present value of profits or the net liability equals zero.

No-arbitrage pricing and actuarial pricing practices are derived from entirely different theoretical basis. Actuarial pricing practice is based on the quantile of the net liability under the real world measure. The no-arbitage pricing is based on the expectation of the net liability under the risk-neutral measure.

(ii) No-arbitrage pricing is not directly used in the insurance industry as

- The principle of no-arbitrage pricing does not explicitly allow for expenses and profits
- No-arbitrage pricing is based on the assumption that the underlying assets and financial derivatives are freely tradable and short-selling is allowed
- Third-party fund manangers manage the assets and thought to develop a replicating portfolio
- Short-selling of investment guarantees are not possible

15. Learning Objectives:

5. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(5b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

(5c) Demonstrate an understanding of dynamic and static hedging for embedded guarantees, including:
- (i) Risks that can be hedged, including equity, interest rate, volatility and cross Greeks.
- (ii) Risks that can only be partially hedged or cannot be hedged including policyholder behavior, mortality and lapse, basis risk, counterparty exposure, foreign bonds and equities, correlation and operation failures

(5d) Demonstrate an understanding of target volatility funds and their effect on guarantee cost and risk control.

Sources:

QFIQ-124-20: Variable Annuity Volatility Management: An Era of Risk-Control

QFIQ-135-22: Structured Product Based Variable Annuities, Deng, Dulaney, Husson, McCann (sections 2 & 3)

Commentary on Question:

Part a: Answers that did not receive full credit often missed the underlying zero-coupon bond asset in constructing the structured portfolio.
Part b/d: Successful candidates understood Black-Scholes option pricing formula and Vega formula
Part c: Candidates often listed strategies without objective
Part e: Most candidates received partial credit on allocations and target vol fund purpose from recall

Solution:

(a) Assume an initial deposit of \$100 is made to this spVA product when the reference asset value is \$100.

(i) Express the buffer level and the cap level, in terms of B% and C%.

(ii) Sketch the maturity payoff of this spVA against the reference asset value (label the buffer and cap levels).

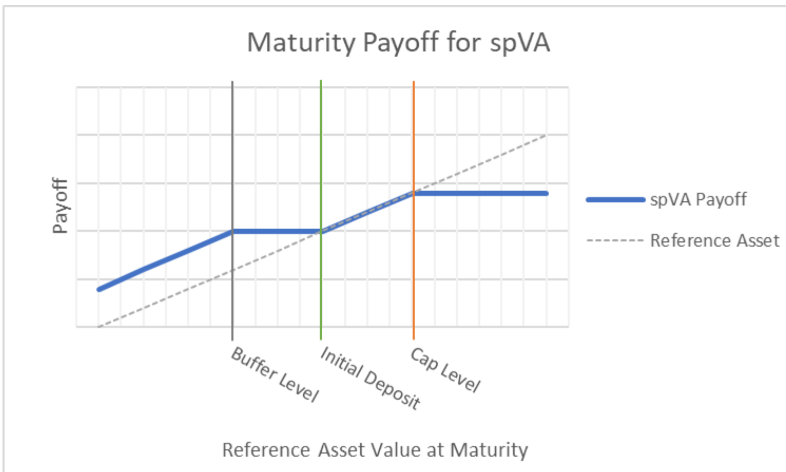
(iii) Specify and justify a portfolio of bonds and options that replicates the maturity payoff of the spVA.

15. Continued

Commentary on Question:
Full credit was given to part (iii) as long as all four assets are listed without formulas

i) Cap Level = (1+ C%) *Initial Deposit of $100
The maximum return that issuer pays to investors when there is positive return at maturity.

Buffer Level = (1- B%) *Initial Deposit of $100
The level that investors start to be exposed to losses (i.e., If the value of reference asset falls below this level, investors would start to incur losses).



iii)
Buffered and capped structured products can be decomposed into a portfolio of four assets:

- a zero-coupon bond with a face amount equal to initial deposit of $100 ------ (1)
- a short European put option with a strike price at the buffer level

$-P$ (Buffer) = – Max (Buffer – S_t, 0)
= – [Buffer – S_t | if S_t < Buffer] -------(2)

- a long European at-the-money call option

C (100) = Max (S_t – 100 0)
= [S_t - 100 | if S_t > 100] -------(3)

- a short European call option with a strike price at the cap level.

–C(Cap) = – Max (S_t – Cap, 0)
= – [S_t - Cap | if S_t >Cap] -------(4)

Summing up (1) to (4)
Payoff = [S_t – Buffer +100 | if S_t, < Buffer]
+ [100 | if 100 > S_t > Buffer]
+ [S_t | if Cap > S_t > 100]
+ [Cap | if S_t > Cap] ← match the payoff graph in (ii)

15. Continued

(b) Calculate and determine whether the fair cap level is 5.5%, 7.5% or 10.5% for this spVA product to address the market risk.

Commentary on Question:
Partial credit was given to candidates who demonstrated command of BS formula and was able to calculate value of calls and puts. Candidates often missed the correct answer due to missing ZCB value.

Set Initial Deposit = PV of [Initial Deposit * $\Pi_{i=1 \text{ to } t}$ E(1+R_i)], where

$E(1+R_i)$
$= 1 - (FV_{put}(S_0, K_{buffer}, r, q, \sigma, \tau)/S_0)$
$+ (FV_{call}(S_0, K= S_0, r, q, \sigma, \tau)/S_0) - (FV_{call}(S_0, K_{cap}, r, q, \sigma, \tau)/S_0)$

$FV_{put}(S_0 = 100, K, r=2\%, q=0\%, \sigma=10\%, \tau=1)$
$= KN(-d_2) - 100^* e^{2\%} * N(-d_1)$

$FV_{Call}(S_0 = 100, K, r=2\%, q=0\%, \sigma=10\%, \tau=1)$
$= 100^* e^{2\%} * N(d_1) - KN(d_2)$

, where $d_1 = [\ln(100/K) + 2\% + 0.5 * 10\%\text{^}2]/10\%$
$d_2 = d_1 - 10\%$

$FV_{put}(S_0 = 100, K_{buffer}=90, r=2\%, q=0\%, \sigma=10\%, \tau=1) = 0.48068$
$FV_{call}(S_0 = 100, K_{atm}=100, r=2\%, q=0\%, \sigma=10\%, \tau=1) = 5.11833$

100
$= e^{-2\%(1)}\ 100 * [1 - (FV_{put}(S_0, K_{buffer}, r, q, \sigma, \tau)/S_0)$
$+ (FV_{call}(S_0, K= S_0, r, q, \sigma, \tau)/S_0) - (FV_{call}(S_0, K_{cap}, r, q, \sigma, \tau)/S_0)]$
➔
$1 = e^{-2\%(1)} * [1 - (FV_{put}(S_0, K_{buffer}, r, q, \sigma, \tau)/S_0)$
$+ (FV_{call}(S_0, K= S_0, r, q, \sigma, \tau)/S_0) - (FV_{call}(S_0, K_{cap}, r, q, \sigma, \tau)/S_0)]$

$= e^{-2\%} * [1-(0.00481)+(0.05118)-(FV_{call}(S_0, K_{cap}, r, q, \sigma,\tau)/S_0)]$

➔ $FV_{call}(S_0, K_{cap}, r, q, \sigma,\tau)/S_0 = 1-(0.00481)+(0.05118) - e^{2\%}$
$= 0.0262$

➔ $K_{cap} = 105.53$ → cap rate = 5.53%

15. Continued

(c) List four key metrics for evaluating risk management strategy, together with their objectives.

Commentary on Question:
Full credit was only given when strategies are listed with corresponding objectives; Full credit was given when 4 out of 5 metrics was provided.

- Guaranteed Cost: PV of fee – PV of guarantee claims

Objective: Write profitable business. Do the risk-controls reduce the hedge cost of the guarantees?
Measure whether the business is profitable with the hedge program.

- Vega: change in guaranteed cost per 1% change in volatility.

Objective: Stablize ALM and hedging performance. Do the volatility management strategies improve Vega?
Measure whether the performance of hedge program is stable, despite of volatility fluctuation.

- Hedge Performance: Hedge P&L

Objective: Stablize ALM and hedging performance. How well do the risk-control strategies minimize hedge P&L losses in crises?
Measure whether the hedge program results in stable P&L over time, despite market fluctuations. (ie. equity, interest rate)

- Basis Risk: effect of tracking error produced by imperfect knowledge of investment positions

Objective: Stablize ALM and hedging performance. Can the risk management effectively mirror the changing fund positions?

- Reserve Impact & Volatility: portfolio values in "tail" of the distribution

Objective: Optimize capital requirements. Do the funds reduce Statutory reserve requirements (and volatility of reserves)?

(d)

(i) Explain how the spVA sales could be used to mitigate the market risk of inforce business, in terms of the embedded option and pricing control through the cap.

(ii) Calculate the total Vega for QFI Life's business, including the \$50M of new spVA sales.

(iii) Calculate the change in Vega, assuming implied volatility suddenly increases to 20%.

(iv) Assess the impact of the spVA business on the change in Vega, based on results in part (ii) and part (iii).

15. Continued

Commentary on Question:
Candidates often missed the natural hedge of spVA short position with inforce VA long put position.

(i)
spVA has short position in Put (P(90)), while Inforce VA business has long position in Put (P(90)). The spVA sales naturally offset some market risk exposure with Inforce VA business under downward market.

The design of spVA include a cap, determining the upside return that would be credited to the investor at maturity. spVAs yield the issuer a tremendous degree of pricing control through the cap, which is often priced in premium for the market risks with a lower than fair value cap.

(ii) &(iii)
For Inforce VA Business:
Embedded option: P(90)
# of unit: \$100M/ \$100 = 1M

Vega for P(90) w/ implied volatility @ 10%
= SN'(d1) sqrt(T-t) = 100*N'(1.3) sqrt(1) = 17.057
- ➔ Long 1M unit * 17.057 = 17.057M

Vega for P(90) w/ implied volatility @ 20%
= SN'(d1) sqrt(T-t) = 100*N'(0.73) sqrt(1) = 30.63
- ➔ Long 1M unit * 30.63 = 30.63M

For new sales in spVA:
Embedded option: – P(90) + C(100) – C(105.53)
# of unit: \$50M/ \$100 = 0.5M

Vega for P(90) w/ implied volatility @ 10%
= SN'(d1) sqrt(T-t) = 100*N'(1.30) sqrt(1) = 17.057
- ➔ Short 0.5M unit * 17.057= – 8.5283M

Vega for P(90) w/ implied volatility @ 20%
= SN'(d1) sqrt(T-t) = 100*N'(0.73) sqrt(1) = 30.63
- ➔ Short 0.5M unit * 30.63= – 15.3170M

Vega for C(100) w/ implied volatility @ 10%
= SN'(d1) sqrt(T-t) = 100*N'(0.25) sqrt(1) = 38.667
- ➔ Long 0.5M unit * 38.667= 19.3334M

15. **Continued**

Vega for C(100) w/ implied volatility @ 20%
= SN'(d1) sqrt(T-t) = 100*N'(0.2) sqrt(1) = 39.104

- ➔ Long 0.5M unit * 39.104= 19.5521M

Vega for C(105.53) w/ implied volatility @ 10%
= SN'(d1) sqrt(T-t) = 100*N'(-0.29) sqrt(1) = 38.2708

- ➔ Short 0.5M unit * 38.2708= -19.1354M

Vega for C(105.53) w/ implied volatility @ 20%
= SN'(d1) sqrt(T-t) = 100*N'(-0.07) sqrt(1) = 39.799

- ➔ Short 0.5M unit * 39.799= – 19.8995M

Vega for QFI Life's business w/ volatility @ 10%
= 17.057M – 8.5283M +19.3334M – 19.1354M= 8.7263M

Vega for QFI Life's business w/ volatility @ 20%
= 30.63M – 15.317M +19.5521M – 19.8995M = 14.9696M

- ➔ Change in Vega = 14.9696M– 8.7263M= 6.2433M

(iv)

Without the $50M spVA business, the change in the vega of the VA businss is 30.63M - 17.057M = 13.5774M.

So, the $50M spVA business <u>reduces</u> the change in vega by 54% (=*6.2433M/*13.5774M - 1).

(v)
Hedge Inforce VA with new sales in spVA

- Reduced exposure to the downside market risks.
- spVA's short position in P(90) reduces the volatility exposure with Inforce VA, which has a long position in P(90). The reduction in volatility exposure is 54% because the spVA new sales are 50% of the inforce VA business.
- spVA's long position in C(100) and short position in C(105.53) results in a net negative Vega exposure at the higher volatility level (20%), slightly offseting the positive Vega exposure with Inforce VA. However, it doesn't provide stable hedge to the overall Vega exposure. Depends on the level of implied volatility, spVA's call options could increase the exposure to volatility change (e.g., when the volatility is 10%).

15. Continued

(e) Under Scenario 1 and Scenario 2, respectively

(i) Determine the allocation to the underlying asset.

(ii) Calculate the total Vega of QFI Life's VA business (i.e., no spVA).

(iii) Explain how the target volatility fund reduces the market risk of inforce business, based on results in part (i) and part (ii).

(i)

Equity Ratio = minimum (110%, 10%/ realized equity volatility)

Scenario 1:
Equity Ratio = Min(110%, 10%/10%) = 100%

Scenario 2:
Equity Ratio = Min(110%, 10%/20%) = 50%

(ii)
Embeded option: P(90)

Scenario 1:
Implied volatility = 100% * 10% = 10%

Scenario 2:
Implied volatility = 50% * 20% = 10%

Under both Scenario 1 and Scenario 2:

Vega for QFI's VA business
= Vega for P(90)
= $SN'(d_1)$ sqrt(T-t) = 100*N'(1.30) sqrt(1) = *17.057M*

(iii) Transfer all deposits to target volatility fund under VA

- Actively manage the volatility exposure
- In the case of a volatility spike (ie. 10% ->20%), vega can stay unchanged, as the fund volatility is managed at 10%
- However, the business is still subject to vega change when volatility drops below 9%, as the maximum allocation is capped at 110%