

# GI ADV Model Solutions

## Fall 2022

### 1. Learning Objectives:

4. The candidate will understand how to apply the fundamental techniques of reinsurance pricing.

### Learning Outcomes:

- (4c) Calculate the price for a casualty per occurrence excess treaty.

### Sources:

Basics of Reinsurance Pricing, Clark

### Commentary on Question:

*This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.*

### Solution:

- (a) Calculate the loss cost rate for the treaty.
  1. Calculate the ELF at the bottom of the layer (250,000) and the top of the layer (750,000) for each state/hazard group using the NCCI ELF parameters for the appropriate hazard group.
  2. Calculate the layer loss cost for each state/hazard group as the ELF difference (bottom of the layer minus top of the layer) times the standard premium for the state/hazard group times the expected loss ratio for the state.
  3. Calculate the loss cost rate for the treaty as the total loss cost divided by the total standard premium. This equals 2.52%.
- (b) Describe the characteristics of the following categories:
  - (i) Working Layer
  - (ii) Exposed Excess
  - (iii) Clash Cover

### Commentary on Question:

*The model solution is an example of a full credit solution.*

## 1. Continued

- (i) Working Layer: Layer that regularly has claims over the experience period.
  - (ii) Exposed Excess: Layer likely to have a small number of claims over the experience period.
  - (iii) Clash Cover: Layer may have only a few claims, or no claims over the experience period.
- (c) Compare the use of experience rating in pricing treaties in the three categories.

### **Commentary on Question:**

*The model solution is an example of a full credit solution.*

- (i) Working Layer: Experience rating is generally used and is likely to give a rate representative of the risk.
- (ii) Exposed Excess: Experience rating is of limited usefulness.
- (iii) Clash Cover: Experience rating is generally not useful.

## 2. Learning Objectives:

5. The candidate will understand methodologies for determining an underwriting profit margin.

### Learning Outcomes:

- (5b) Calculate an underwriting profit margin using the capital asset pricing model.

### Sources:

Ratemaking: A Financial Economics Approach, D'Arcy and Dyer

### Commentary on Question:

*This question required the candidate to respond in Excel for parts (b), (c) and (d). An example of a full credit solution for these parts is in the Excel solutions spreadsheet. The solution in this file for parts (b), (c) and (d) is for explanatory purposes only.*

### Solution:

- (a) Describe two methods for obtaining the underwriting beta.

Method 1: Perform a regression of historical underwriting returns against the returns of the market portfolio.

Method 2: Determine the liability beta and then multiply it by  $-k$ , where  $k$  is the funds generating coefficient.

- (b) Calculate the funds generating coefficient estimate,  $k$ .

The estimate of  $k$  is the sum, over the payment pattern groups, of the product of the percentage of business in the group times the average time between receipt of premium and payment of losses and expenses.  $k = 1.03$ .

- (c) Calculate the UPM ignoring taxes.

$$\text{UPM} = -k \times \text{risk-free rate} + \text{underwriting beta} \times (\text{expected return on the market portfolio} - \text{risk-free rate}) = -0.86\%$$

- (d) Calculate the UPM accounting for taxes.

Average tax rate on investment income (TA) equals 16% which is the sum, over the asset types, of the product of the percentage of total assets in the asset type times the tax rate for the asset type.

$$\text{UPM} = -k \times \text{risk-free rate} \times (1 - \text{TA}) / (1 - \text{tax rate on underwriting income}) + \text{underwriting beta} \times (\text{expected return on the market portfolio} - \text{risk-free rate}) + (1 / \text{ratio of premium to owner's equity}) \times \text{risk-free rate} \times \text{TA} / (1 - \text{tax rate on underwriting income}) = -0.89\%$$

### 3. Learning Objectives:

1. The candidate will understand how to use basic loss development models to estimate the standard deviation of an estimator of unpaid claims.

#### Learning Outcomes:

- (1e) Apply a parametric model of loss development.
- (1f) Estimate the standard deviation of a parametric estimator of unpaid claims.

#### Sources:

LDF Curve Fitting and Stochastic Reserving: A Maximum Likelihood Approach, Clark

#### Commentary on Question:

*This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.*

#### Solution:

- (a) Calculate the MLEs of  $ULT$  for each accident year.
  1. For each row of data provided, calculate  $G$ (at the end of the interval) and  $G$ (at the beginning of the interval).
    - The end of the interval is “To” months minus 6.
    - The beginning of the interval is “From” months minus 6 if “From” months  $\geq 12$  and 0 if “From” months is 0.
  2. The  $MLE_{ULT}$  for each accident year (AY) is the AY total divided by  $G$ (at the end of the interval in which the AY is as of year-end 2021).
- (b) Calculate the value of the loglikelihood function at its maximum.
  1. For each row of data provided, calculate the expected increment which is equal to the  $MLE_{ULT}$  of the accident year times [ $G$ (at the end of the interval) minus  $G$ (at the beginning of the interval)].
  2. Then, for each row of data provided, calculate the incremental payment times the natural log of the expected increment minus the expected increment.
  3. The value of the loglikelihood function at its maximum is the sum of the amounts in 2 which is equal to 169,391.727.
- (c) Estimate  $\sigma^2$ , the scale factor.
  1. For each row of data provided, calculate [the square of the increment minus the expected increment] divided by the expected increment.
  2. Then, the scale factor equals the sum of the amounts in 1 divided by [the number of data points minus the number of parameters estimated] which is equal to 138.7457.

- (d) Explain why an accident year with a single incremental value will always contribute zero to the estimate.

**Commentary on Question:**

*The model solution is an example of a full credit solution.*

Let  $c$  be the incremental loss and  $G$  be the value of the distribution function at the end of the interval. The estimate of ULT is  $c/G$ . The estimated expected increment,  $\mu$ , is  $ULT \times G = c$ . Hence, the estimate always matches the observed increment and the difference must be zero.

- (e) Estimate the process variance of the reserve for accident year 2020.

Process variance of the reserve for accident year 2020 equals scale factor from part (c) times MLE\_ULT for AY 2020 times  $[1 - G(\text{at the end of the interval in which AY 2020 is as of year-end 2021, i.e., at 18 months})]$  which equals 116,603.341.

- (f) Estimate the parameter variance of the reserve for accident year 2020.

- The formula for the reserve is MLE\_ULT times  $(1 - G)$ . The derivative with respect to MLE\_ULT is  $(1 - G)$  and the derivative with respect to theta is  $- \text{MLE\_ULT times } G'$ .  $G' = - (1 - G(\text{at the end of the interval}))$  times  $(\text{"To" months} - 6)$  divided by the square of the MLE of theta.
- Let  $\text{COV}(x,y)$  be the covariance matrix (*provided in the Excel file*).
- For AY 2020:
  - $(1 - G(\text{AY 2020 at 18 months})) = 0.11$ .
  - $- \text{MLE\_ULT}_{2020} \text{ times } G' = - 7,640.411 \times - 0.02977 = 227.4825$ .
  - Parameter variance of the reserve for AY 2020 equals  $[(1 - G)^2 \times \text{COV}(\text{ULT}_{2020}, \text{ULT}_{2020})]$   
 $+ [2 \times (1 - G) \times (- \text{MLE\_ULT}_{2020} \times G') \times \text{COV}(\text{theta}, \text{ULT}_{2020})]$   
 $+ [(- \text{MLE\_ULT}_{2020} \times G')^2 \times \text{COV}(\text{theta}, \text{theta})]$   
 $= 80,094.703$ .

#### 4. Learning Objectives:

1. The candidate will understand how to use basic loss development models to estimate the standard deviation of an estimator of unpaid claims.

#### Learning Outcomes:

- (1b) Test for the validity of these assumptions.
- (1c) Identify alternative models that should be considered depending on the results of the tests.
- (1d) Estimate the standard deviation of a chain ladder estimator of unpaid claims.

#### Sources:

Measuring the Variability of Chain Ladder Reserve Estimates, Mack

Testing the Assumptions of Age-to-Age Factors, Venter

#### Commentary on Question:

*This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.*

#### Solution:

- (a) Demonstrate that the reserve and standard error for accident year 5 have been correctly calculated.

Reserve for accident year (AY) 5:

- Given as 7,221.
- Need to estimate the projected payments for AY5. Use the appropriate value of  $f_k$  and the preceding cumulative payment to get a projected payment. We then get the cumulative projected payments at development years 4 through 7 as follows: 12,962, 14,649, 16,167 and 16,879. The reserve for AY5 is 16,879 minus 9,658 which equals 7,221.

Standard error for AY5:

- Given as 1,592.
- Estimate as the square root of:  
$$c(5, 7)^2 \times [\sigma_3^2 / f_3^2 \times (1 / c(5,3) + 1 / (c(1,3) + c(2,3) + c(3,3) + c(4,3)))$$
$$+ \sigma_4^2 / f_4^2 \times (1 / c(5,4) + 1 / (c(1,4) + c(2,4) + c(3,4)))$$
$$+ \sigma_5^2 / f_5^2 \times (1 / c(5,5) + 1 / (c(1,5) + c(2,5)))$$
$$+ \sigma_6^2 / f_6^2 \times (1 / c(5,6) + 1 / c(1,6))]$$
- This equals 1,592.

## 4. Continued

- (b) Estimate the coefficient of variation of the unpaid claims for each of accident years 2-7 and overall.

The coefficient of variation (CoV) estimate is given by the standard error estimate divided by the reserve estimate. This is done for AYs 2 to 7 and overall.

- (c) Estimate the upper 90% confidence limit of the overall unpaid claims using a lognormal distribution. (Note: The 90<sup>th</sup> percentile of a standard normal distribution is 1.28.)

This is equal to the total reserve estimate times the exponential of 1.28 times sigma for the total minus sigma-squared for the total divided by 2. Sigma-squared for the total is the natural log of (1 plus the square of the CoV for the total). The upper 90% confidence limit of the overall unpaid claims using a lognormal distribution is 57,402.

- (d) Allocate the overall amount from part (c) to accident years 2-7 in such a way to reach the same level of confidence for each accident year. (Note: Using Excel's Goal Seek function is an acceptable approach.)

### Commentary on Question:

*The model solution in Excel used Excel's Goal Seek function.*

First, one needs to calculate the upper 90% confidence limit of the unpaid claims for each AY. These are calculated by using the approach from part (c), except using AY values instead of total values. The Goal Seek "set cell" has the adjusted confidence limit sum minus the amount from (c). The adjusted confidence limit for each AY is the confidence limit formula with 1.28 replaced by a cell that changes in Goal Seek (using any value close to 1.28 for the initial value in the change cell works). This gives the following:

Accident Year	Adjusted Confidence Limit
2	1,047
3	3,133
4	6,105
5	8,476
6	13,573
7	25,067
Overall	57,402

## 4. Continued

- (e) Explain the empirical approach to establishing confidence limits as described by Mack.

**Commentary on Question:**

*The model solution is an example of a full credit solution.*

Use the minimum and maximum individual age-to-age factors in each column to establish minimum and maximum limits.

- (f) Explain why this empirical approach to establishing confidence limits does not seem to be reasonable.

**Commentary on Question:**

*The model solution is an example of a full credit solution.*

The different number of age-to-age factors in each column results in inconsistent confidence limits among accident years.

- (g) Create a scatter plot to check the assumption that the expected losses at age 2 are proportional to the losses at age 1.

For AYs 1 to 6, select the  $x$ -values to be the AY payments at development year 1 and the  $y$ -values to be the AY payments at development year 2.

- (h) Interpret the scatter plot in part (g) with regard to determining whether the assumption is correct. If it is not, recommend an alternative model.

**Commentary on Question:**

*The model solution is an example of a full credit solution.*

The intercept of a line through the points is well above zero, so the assumption does not appear to be correct. A constant term should be added to the model.



## 5. Learning Objectives:

5. The candidate will understand methodologies for determining an underwriting profit margin.

### Learning Outcomes:

- (5d) Allocate an underwriting profit margin (risk load) among different accounts.

### Sources:

An Application of Game Theory: Property Catastrophe Risk Load, Mango

### Solution:

- (a) Calculate the risk load multiplier using Kreps' formula.

$y$  = return on margin surplus, selected as 20%

$z$  = standard normal multiplier, selected as 2.0

The risk load multiplier using Kreps' formula is  $yz / (1 + y) = 0.2 \times 2 / 1.2 = 1/3$ .

- (b) Describe how Mango converted this multiplier for use in the Marginal Variance method to ensure that the two methods have the same total risk load for the portfolio.

This multiplier is divided by the standard deviation of the portfolio.

- (c) Mango refers to portfolio variance as a *super-additive characteristic function*.

Explain what is meant by this reference.

Because of the covariance component, the variance of a portfolio is greater than the sum of the individual account variances.

- (d) Compare the Shapley Value under a variance-based method to the Marginal Variance for calculating a risk load when adding a new account to an existing portfolio.

Shapley Value = Variance(n) + Covariance(L,n)

Marginal Variance = Variance(n) + 2×Covariance(L,n)

Difference is Covariance(L,n)

Marginal Variance > Shapley Value if the covariance of L and n is > 0

- (e) Explain why Mango did not pursue the use of a Shapley Value under a standard deviation-based method.

This is due to the complex nature of the mathematics. There is no simplifying reduction formula as there exists when using the variance.

## 6. Learning Objectives:

2. The candidate will understand the considerations in selecting a risk margin for unpaid claims.

### Learning Outcomes:

- (2c) Describe methods to assess this uncertainty.

### Sources:

A Framework for Assessing Risk Margins, Marshall, et al.

### Commentary on Question:

*This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only.*

### Solution:

- (a) Calculate the internal systemic risk CoV for each of the following:

- (i) Motor CL
- (ii) Home CL
- (iii) Total CL

For (i) and (ii), the first step is to know which source of risk is associated with each of the 6 potential risk indicators. 1 is parameter risk, 2 is data risk, 3 is specification risk, 4 is specification risk, 5 is data risk and 6 is parameter risk. Given the weights of 25% for specification risk, 55% for parameter risk and 20% for data risk and the fact that risk indicator weights are equal within each source of internal systemic risk (ISR), we have the following weights by potential risk indicator: 27.5% for each of 1 and 6, 10% for each of 2 and 5 and 12.5% for each of 3 and 4. This gives a weighted average score of 4.6 for Motor CL and 3.35 for Home CL. Using the scorecard table, this produces an ISR\_CoV of 7% for Motor CL and 7.5% for Home CL. For (iii), we note that the percentage of CL in Motor is  $5,351 / (5,351 + 2,486) = 68.3\%$  ( $P_M$ ) and percentage of CL in Home is  $1 - 68.3\% = 31.7\%$  ( $P_H$ ). Then the Total CL internal systemic risk CoV is given by:  $[ISR\_CoV_M^2 P_M^2 + ISR\_CoV_H^2 P_H^2 + 2 \times \text{Correlation}(\text{Motor CL}, \text{Home CL}) \times ISR\_CoV_M ISR\_CoV_H P_M P_H]^{0.5} = 6.04\%$ .

## 6. Continued

- (b) Calculate the internal systemic risk CoV for total insurance liabilities, both lines combined.

This is the square root of the sum of the product of the following 3 matrices:

1. internal systemic risk correlation matrix of i and j
2. the  $CoV_i$  times  $CoV_j$  matrix
3. liability weight (W) matrix of  $W_i$  time  $W_j$

Matrix 1 is provided.

Matrix 2 is created from the ISR CoVs.

Matrix 3 is created from the weights by liability.

This results in the ISR\_CoV for total insurance liabilities, both lines combined, being equal to 5.01%.

- (c) Calculate the risk margin for the total insurance liabilities at the 75% adequacy level.

### **Commentary on Question:**

*The model solution uses the exact value of the z-value (0.67448975..). It was also acceptable to use the approximate value of 0.674.*

This is equal to:

$$\begin{aligned} & z\text{-value} \times \text{the total insurance liabilities} \\ & \times [\text{CoV Independent Risk squared} + \text{CoV External Systemic Risk squared} + \\ & \quad \text{CoV Internal Systemic Risk squared} + 2 \times 20\% \times \text{CoV External Systemic} \\ & \quad \text{Risk} \times \text{CoV Internal Systemic Risk}]^{0.5} \\ & = z\text{-value} \times 16,045,000 \times 0.112089 \\ & = 1,213,047 \end{aligned}$$

**7. Learning Objectives:**

3. The candidate will understand excess of loss coverages and retrospective rating.

**Learning Outcomes:**

(3d) Explain retrospective rating in graphical terms.

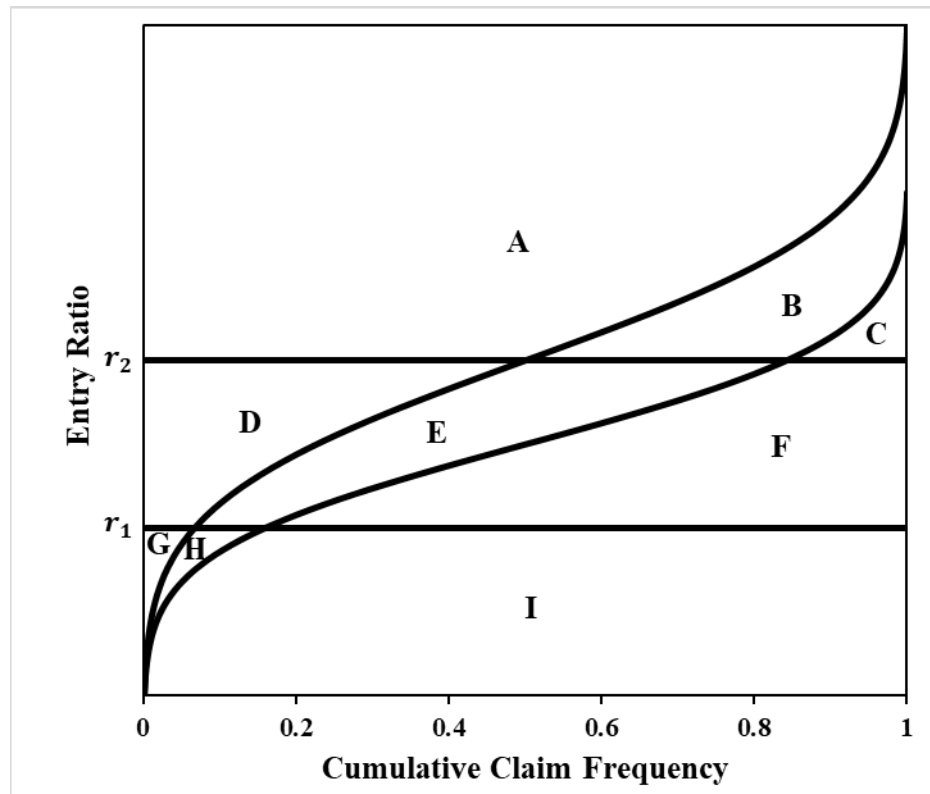
(3e) Explain Table M and Table L construction in graphical terms.

**Sources:**

The Mathematics of Excess of Loss Coverages and Retrospective Rating – A Graphical Approach, Lee

**Solution:**

You are given the following graph of the loss distribution for a risk with a retrospective rating plan, both with and without a per accident loss limitation:



Express the following quantities using the labels for the nine areas on the graph:

## 7. Continued

- (i) 1
- (ii)  $k$ , the loss elimination ratio
- (iii)  $\psi(r_1)$ , the Table M savings
- (iv)  $\psi^*(r_1)$ , the Table L savings
- (v)  $\phi(r_2)$ , the Table M charge
- (vi)  $\phi^*(r_2)$ , the Table L charge
- (vii)  $E\{L\}/E$ , the ratio of the expected limited loss to the expected unlimited loss without a per accident loss limitation
- (viii)  $E\{L^*\}/E$ , the ratio of the expected limited loss to the expected unlimited loss with a per accident loss limitation

- (i)  $1 = B + C + E + F + H + I$
- (ii)  $k = B + E + H$
- (iii)  $\psi(r_1) = G$
- (iv)  $\psi^*(r_1) = G + H$
- (v)  $\phi(r_2) = B + C$
- (vi)  $\phi^*(r_2) = B + C + E + H$
- (vii)  $E\{L\}/E = E + F + G + H + I$
- (viii)  $E\{L^*\}/E = F + G + H + I$

## 8. Learning Objectives:

4. The candidate will understand how to apply the fundamental techniques of reinsurance pricing.

### Learning Outcomes:

- (4d) Apply an aggregate distribution model to a reinsurance pricing scenario.

### Sources:

Basics of Reinsurance Pricing, Clark

### Commentary on Question:

*This question required the candidate to respond in Excel. An example of a full credit solution is in the Excel solutions spreadsheet. The solution in this file is for explanatory purposes only. This solution uses “B” to denote billion.*

### Solution:

- (a) Describe what is meant by “collective risk model.”

Separate modeling of the claim frequency distribution and the severity distribution and then they are combined to produce the aggregate loss distribution.

- (b) Calculate the mean and coefficient of variation for the aggregate losses.

Mean

$$\begin{aligned} &= \text{count mean} \times \text{severity mean} \\ &= 1.5 \times [1B \times 0.6 + 2B \times 0.3 + 3B \times 0.1] = 2.25B \end{aligned}$$

Coefficient of variation

$$\begin{aligned} &= [1B^2 \times 0.6 + 2B^2 \times 0.3 + 3B^2 \times 0.1]^{0.5} / \text{Mean} \\ &= 0.8944 \end{aligned}$$

## 8. Continued

(c) Complete the following aggregate loss probability table:

Aggregate Losses (billions)	Probability
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	

At 0 aggregate losses, the probability is given by the Poisson formula with mean 1.5 for 0 losses.  $P(0) = 1.5^0 \times e^{-1.5} / 0! = 0.2231$

For amounts of aggregate losses (where values are integer amounts in B), we use the recursion formula for the remaining values of X in the table. The recursion formula is as follows:

$$P(X) = 1.5 \times (P(X - 1) \times 0.6 + 2 \times P(X - 2) \times 0.3 + 3 \times P(X - 3) \times 0.1) / X$$

So, we have:

$$P(1) = 1.5(0.2231 \times 0.6)/1 = 0.2008$$

$$P(2) = 1.5(0.2008 \times 0.6 + 2 \times 0.2231 \times 0.3) / 2 = 0.1908$$

$$P(3) = 1.5(0.1908 \times 0.6 + 2 \times 0.2008 \times 0.3 + 3 \times 0.2231 \times 0.1) / 3 = 0.1509$$

$$P(4) = 1.5(0.1509 \times 0.6 + 2 \times 0.1908 \times 0.3 + 3 \times 0.2008 \times 0.1) / 4 = 0.0995$$

And so on, to complete the table.

## 8. Continued

<b>Aggregate Losses (billions)</b>	<b>Probability</b>
0	0.2231
1	0.2008
2	0.1908
3	0.1509
4	0.0995
5	0.0622
6	0.0356
7	0.0190
8	0.0096
9	0.0046
10	0.0021
11	0.0009
12	0.0004
13	0.0002
14	0.0001
15	0.0000