QFI QF Model Solutions
Spring 2023

1. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1a) Understand and apply concepts of probability and statistics important in mathematical finance.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014

Commentary on Question:

This question tests candidates' understanding of basic concepts in probability and mathematical finance. Most candidates were able to earn partial credits of this questions. However, very few candidates were able to earn full credits.

Solution:

(a) Calculate $P(S_T - S_0 = 2m - k)$.

Commentary on Question:

Most candidates were able to derive the probability by counting the paths and adding the probabilities of these paths.

The probability $P(S_T - S_0 = 2m - k)$ is the probability that of the k incremental changes observed, m would be made of +1's and $k - m$ made of -1's.

As a result, the probablity can be obtained by adding all probabilities associated with these combinations:

$$P(S_T - S_0 = 2m - k) = C_k^{k-m} p^m (1-p)^{k-m} = \frac{k!}{m!(k-m)!} p^m (1-p)^{k-m}$$

(b) Determine p such that $\{S_{t_i}\}$ is a martingale with respect to the information set generated by the past price changes and the probability distribution P.

1. Continued

Commentary on Question:
Most candidates calculated the value correctly.

Note that

$$S_{t_i} = S_0 + \Delta S_{t_1} + \cdots + \Delta S_{t_i}$$

We have

$$E\left[S_{t_i} | S_0, \Delta S_{t_1}, \cdots, \Delta S_{t_{i-1}}\right] = S_0 + \Delta S_{t_1} + \cdots + \Delta S_{t_{i-1}} + E\left[\Delta S_{t_i}\right] = S_{t_{i-1}} + E\left[\Delta S_{t_i}\right]$$
$$= S_{t_{i-1}} + p - (1-p) = S_{t_{i-1}} + 2p - 1$$

For $\{S_{t_i}\}$ to be a martingale, we need

$$E\left[S_{t_i} | S_0, \Delta S_{t_1}, \cdots, \Delta S_{t_{i-1}}\right] = S_{t_{i-1}}$$

Combine the above two equations, we get

$$S_{t_{i-1}} + 2p - 1 = S_{t_{i-1}}$$

Which gives $p = \frac{1}{2}$

(c) Express $E^P[S_T | I_1]$ and $E^P[S_T | I_3]$ in terms of S_{t_1} and S_{t_3}, respectively.

Commentary on Question:
Many candidates did this part correctly.

By the assumptions, we have

$$E[S_T | I_1] = E\left[1 + \Delta S_{t_1} + \Delta S_{t_2} + \Delta S_{t_3} + \Delta S_{t_4} | I_1\right] = 1 + \Delta S_{t_1} + E\left[\Delta S_{t_2} + \Delta S_{t_3} + \Delta S_{t_4} | I_1\right]$$
$$= S_{t_1} + E\left[\Delta S_{t_2} + \Delta S_{t_3} + \Delta S_{t_4}\right] = S_{t_1} + 3(2p-1) = S_{t_1} + 1.5$$

Similarly, we have

$$E[S_T | I_3] = E\left[1 + \Delta S_{t_1} + \Delta S_{t_2} + \Delta S_{t_3} + \Delta S_{t_4} | I_3\right] = 1 + \Delta S_{t_1} + \Delta S_{t_2} + \Delta S_{t_3} + E\left[\Delta S_{t_4} | I_3\right]$$
$$= S_{t_3} + E\left[\Delta S_{t_4}\right] = S_{t_3} + (2p-1) = S_{t_3} + 0.5$$

(d) Compute $Q\left(\Delta S_{t_1} = 1\right)$ and $Q\left(\Delta S_{t_1} = -1\right)$.

Commentary on Question:
Many candidates forgot the arbitrage theorem and did not get the correct numbers for this part.

Denote

$$Q\left(\Delta S_{t_1} = 1\right) = q, \quad \text{and} \quad \left(\Delta S_{t_1} = -1\right) = 1 - q$$

From the arbitrage theorem, we know that

$$S_{t_0} = \frac{1}{1+r}\left(q(S_0 + 1) + (1-q)(S_0 - 1)\right)$$

1. **Continued**

Since $r = 0.05$ and $S_0 = 1$, we have

$$1 = \frac{1}{1 + 0.05}(2q)$$

This gives

$$q = 0.525$$

(e) Explain the following tools used for asset pricing:

(i) The martingale representation theorem

(ii) Normalization

(iii) Change of measure

Commentary on Question:
This part is direct information retrieval from the text. However, almost all candidates were not able to fully explain these tools.

(i) Martingale representation theorem. Given a process, we decompose it into a known trend and a martingale.

(ii) Normalization. Normalization helps to eliminate some unwanted terms in the martingale representation theorem.

(iii) Measure change. By calculating the expectations using the risk-neutral probabilities, the remaining unwanted terms in the martingale representation will be vanished.

2. **Learning Objectives:**

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1j) Understand and apply Girsanov's theorem in changing measures.

Sources:

Chin et al, Ch. 4

Commentary on Question:

In general, this part of the question was the one found most accessible for candidates. Most candidates derived part i) correctly. A handful found part ii) difficult. Many used their results in i) to justify ii), despite being asked explicitly to show "using the definition" which means the martingale property must be shown explicitly. Others did not adequately demonstrate the martingale property and mixed up when you could remove arguments outside of the expectation. Many failed to show that E(abs(M(t)) could be simplified to M(0)

Solution:

(a) Show that $M(t)$ is a $\mathbb{Q}$-martingale using each of the following approaches:

(i) Deriving the stochastic dynamics of $M(t)$.

(ii) Applying the definition of a martingale.

(i)

$dM(t) = \alpha M(t)dW(t) - \frac{\alpha^2}{2}M(t)dt + \frac{\alpha^2}{2}M(t)dt$

$= \alpha M(t)dW(t)$

which is driftless and hence $M(t)$ is a martingale under $\mathbb{Q}$.

2. Continued

(ii)
By definition,

$$E^{\mathbb{Q}}(|M(t)|) = E^{\mathbb{Q}}\left(\left|M(0)e^{\alpha W(t)-\frac{\alpha^2 t}{2}}\right|\right)$$
$$= M(0)e^{-\frac{\alpha^2 t}{2}}E^{\mathbb{Q}}\left(e^{\alpha W(t)}\right) = M(0) < \infty$$

Clearly $M(t)$ is $\mathcal{F}_t$-measureable

For $0 < s < t$

$$E_s^{\mathbb{Q}}\big(M(t)\big) = E_s^{\mathbb{Q}}\left(M(0)e^{\alpha W(t)-\frac{\alpha^2 t}{2}}\right)$$
$=M(0)e^{-\frac{\alpha^2 t}{2}}E_s^{\mathbb{Q}}\left(e^{\alpha(W(t)-W(s)+W(s))}\right)$
$=M(0)e^{-\frac{\alpha^2 t}{2}}e^{\alpha W(s)}E_s^{\mathbb{Q}}\left(e^{\alpha(W(t)-W(s))}\right)$, since $W(s)$ is $\mathcal{F}_s$-measureable
$=M(0)e^{-\frac{\alpha^2 t}{2}}e^{\alpha W(s)}e^{\frac{\alpha^2}{2}(t-s)}$, since $W(t)-W(s)$ is normal distribution mean 0 and variance t-s.
$= M(0)e^{\alpha W(s)-\frac{\alpha^2 s}{2}} = M(s)$

Hence by definition, $M(t)$ is a $\mathbb{Q}$-martingale

(b) Write down the Radon-Nikodym derivative of $\mathbb{Q}^A$ with respect to $\mathbb{Q}$.

Commentary on Question:
Of all parts, candidates did the poorest on this section. Most left it blank and/or incorrect. A handful knew the standard definition of the RN derivative but very few successfully provided the correct expression. Another small handful knew the definition but incorrectly derived an expression involving the other asset/numeraire pair and not Q and the risk-free asset.

Candidates could apply a number of approaches to derive the RN-derivative, but since the question states "write-down", they will receive full marks for writing down the correct expression and simplifying to the final answer.

Approach 1 From question a), we have shown $dM(t)$ is driftless under $\mathbb{Q}$ and so is the bank account numeraire, trivially, since r = *0*. $A(t)$ has the equivalent form of $A(t)$ and hence is also driftless under $\mathbb{Q}$

2. Continued

Hence, we can write the RN-derivative as a ratio of asset numeraires:

$$\frac{d\mathbb{Q}^A}{d\mathbb{Q}} = \frac{A(t)/A(0)}{e^{0*t}/e^{0*0}} = e^{\vartheta W(t)-\frac{\vartheta^2 t}{2}}$$

Approach 2 The result follows directly from Girsanov's Theorem since $W(t)$ is a $\mathbb{Q}$-standard Brownian Motion and $-\vartheta$ is constant and therefore $\mathcal{F}_t$-adapted

$$Z_t = e^{-(-\vartheta)W(t)-\frac{(-\vartheta)^2 t}{2}} = e^{\vartheta W(t)-\frac{\vartheta^2 t}{2}} = \frac{d\mathbb{Q}^A}{d\mathbb{Q}}$$

(c) Determine, using Ito's lemma and Girsanov Theorem, whether the normalized process $\frac{M(t)}{A(t)}$ is a $\mathbb{Q}$-martingale or a $\mathbb{Q}^A$-martingale.

Commentary on Question:
Overall, candidates performed fairly well on this question with most recognizing it was not a Q-martingale but was a $\mathbb{Q}^A$-martingale. Some lost marks by not including clear statements that it was not a Q martingale (and instead relied upon the examiner to infer the candidate knew because it had a drift term in the stochastic dynamics it was not a martingale). Some made errors in the Q dynamics but correctly applied Girsanov's theorem to find the dynamics under $\mathbb{Q}^A$ and were not penalized again for the mistake under Q. Some candidates missed full marks as they did not adequately explain that it was Girsanov's theorem that let them incorporate the $\mathbb{Q}^A$ Brownian motion, instead just applying the transformation without any justification.

Let $\widetilde{M}(t) = M(t)/A(t)$. *First simplify the expression and then apply Ito's Lemma*

$$\widetilde{M}(t) = \frac{M(0)e^{\alpha W(t) - \frac{\alpha^2 t}{2}}}{A(0)e^{\vartheta W(t) - \vartheta^2 t/2}} = \widetilde{M}(0)e^{(\alpha-\vartheta)W(t) - \frac{(\alpha^2-\vartheta^2)t}{2}}$$

$$d\widetilde{M}(t) = (\alpha-\vartheta)\widetilde{M}(t)dW(t) - \frac{(\alpha^2-\vartheta^2)}{2}\widetilde{M}(t)dt + \frac{(\alpha-\vartheta)^2}{2}\widetilde{M}(t)dt$$
$$= (\vartheta^2 - \alpha\vartheta)\widetilde{M}(t)dt + (\alpha-\vartheta)\widetilde{M}(t)dW(t)$$

This is not a martingale under $\mathbb{Q}$ *as the dynamics are not driftless*

We have already established in parts a) and b) that
$Z_t = \frac{d\mathbb{Q}^A}{d\mathbb{Q}} = e^{\vartheta W(t)-\frac{\vartheta^2 t}{2}}$ *is a* $\mathbb{Q}$ *martingale*

2. **Continued**

Using Girsanov's theorem, we know there is a $\mathbb{Q}^A$ *standard Brownian Motion, say* $\widetilde{W}(t)$ *such that* $\widetilde{W}(t) = W(t) + (-\vartheta)t$

So, $d\widetilde{M}(t) = (\vartheta^2 - \alpha\vartheta)\widetilde{M}(t)dt + (\alpha - \vartheta)\widetilde{M}(t)\big(d\widetilde{W}(t) + \vartheta dt\big)$
$= \vartheta^2\widetilde{M}(t)dt - \alpha\vartheta\widetilde{M}(t)dt + \alpha\vartheta\widetilde{M}(t)dt - \vartheta^2\widetilde{M}(t)dt + (\alpha - \vartheta)\widetilde{M}(t)d\widetilde{W}(t) = (\alpha - \vartheta)\widetilde{M}(t)d\widetilde{W}(t)$

Hence $\widetilde{M}(t)$ *is driftless and a* $\mathbb{Q}^A$*-martingale*

(d) Derive an expression for today's price of an exchange option with payoff $P(T) = max[0, M(T) - A(T)]$.

Commentary on Question:
More than half of candidates attempt this question; however, few attempts were successful, and overall marks were low on this part. A very small minority of candidates were able to achieve near full marks. There was a very similar (and harder) question on the Spring 2022 paper that would have prepared candidates for this one.

The purpose of this question is to test the application of change of numeraire, either explicitly, or based on the equivalency of asset/numeraire pairs:

$P(0) = E^{\mathbb{Q}}(M(T) - A(T))^+$

This expression cannot easily be evaluated under $\mathbb{Q}$

Rewriting it, $P(0) = E^{\mathbb{Q}}\left(A(T)\frac{(M(T)-A(T))}{A(T)}\right)^+$

$$= E^{\mathbb{Q}}\left(A(T)\big(\widetilde{M}(T) - 1\big)\right)^+$$

$$= E^{\mathbb{Q}}\left(A(0)\frac{A(T)}{A(0)}\big(\widetilde{M}(T) - 1\big)\right)^+$$

$$= A(0)E^{\mathbb{Q}}\left(\frac{d\mathbb{Q}^A}{d\mathbb{Q}}\big(\widetilde{M}(T) - 1\big)\right)^+$$

$$= A(0)E^{\mathbb{Q}^A}\left(\big(\widetilde{M}(T) - 1\big)^+\right)$$

2. **Continued**

But $\widetilde{M}(T)$ *is a* $\mathbb{Q}^A$*-martingale and hence the above expression has the familiar Black-76 formula for a call option struck on* $\widetilde{M}(T)$ *at K=1*

$$P(0) = A(0)\left(\widetilde{M}(0)N(d_1)\right) - N(d_2))$$

$$d_1 = \frac{log\left(\widetilde{M}(0)\right) + \frac{(\alpha - \vartheta)^2 T}{2}}{(\alpha - \vartheta)\sqrt{T}}$$

$$d_2 = d_1 - (\alpha - \vartheta)\sqrt{T}$$

3. Learning Objectives:

1. The candidate will understand the foundations of quantitative finance.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

(1g) Understand the distinction between complete and incomplete markets.

(1h) Define and apply the concepts of martingale, market price of risk and measures in single and multiple state variable contexts.

(1j) Understand and apply Girsanov's theorem in changing measures.

Sources:
Problems and Solutions in Mathematical Finance: Stochastic Calculus, Chin, Eric, Nel, Dian and Olafsson, Sverrir, 2014 (pages 157-159, 241-242)

Solution:

(a) Show that

$$Z_t = \frac{\sigma_S W_t + \sigma_E V_t}{\sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}}$$

is a $\mathbb{P}$-standard Brownian motion.

Commentary on Question:
Candidates generally performed well on this part. Candidates who stated Z_t *satisfied the properties solely due to* W_t *and* V_t *satisfying them only received partial credit as the question directed candidates to "show" the properties are satisfied.*

For Z_t to be a $\mathbb{P}$-standard Brownian motion, it must satisfy:

1. $Z_0 = 0$ and Z_t has continuous sample paths,
2. $Z_t \sim N(0, t)$, and
3. $Z_{t+s} - Z_t \perp Z_t$ for $s > 0$

3. **Continued**

Given that W_t and V_t are standard Brownian motions, we find that:

$$Z_0 = \frac{\sigma_S W_0 + \sigma_E V_0}{\sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}} = \frac{\sigma_S(0) + \sigma_E(0)}{\sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}} = 0$$

Continuous sample paths follow from being a linear combination of two standard Brownian motions.

For second property, Z_t is normal given that it is a linear combination of two normal distributions. However, we must verify the expectation and variance.

Similar to the first property,

$$E[Z_t] = E\left[\frac{\sigma_S W_t + \sigma_E V_t}{\sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}}\right]$$

$$= \frac{1}{\sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}}\{\sigma_S E[W_t] + \sigma_E E[V_t]\} = 0$$

$$Var(Z_t) = E[Z_t^2] = E\left[\left(\frac{\sigma_S W_t + \sigma_E V_t}{\sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}}\right)^2\right]$$

$$= \frac{1}{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E} E[{\sigma_S}^2 W_t^2 + {\sigma_E}^2 V_t^2 + 2\sigma_S W_t \sigma_E V_t]$$

$$= \frac{1}{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}\{{\sigma_S}^2 E[W_t^2] + {\sigma_E}^2 E[V_t^2] + 2\sigma_S\sigma_E E[W_t V_t]\}$$

$$= \frac{1}{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}\{{\sigma_S}^2 t + {\sigma_E}^2 t + 2\sigma_S\sigma_E\rho t\} = t$$

For the final property, given that $Z_{t+s} - Z_t$ and Z_t are jointly normally distributed, it suffices to show $Cov(Z_{t+s} - Z_t\ , Z_t) = E[(Z_{t+l} - Z_t)Z_t] = 0$.

With Z_t being $\mathcal{F}_t$-measurable, $E[(Z_{t+l} - Z_t)Z_t] = E[Z_t E[Z_{t+l} - Z_t|\mathcal{F}_t]] = 0$.

(b) Determine whether ln $(S_t E_t)$ follows an arithmetic Brownian motion under the measure $\mathbb{P}$ or not.

Commentary on Question:
Candidate who attempted to perform Ito's Lemma directly on ln $(S_t E_t)$ *often left out a cross-term. Those who first determined the form of* $d(S_t E_t)$ *or followed the alternate approach were most successful. Some candidates conflated the notion of arithmetic Brownian motion and martingale. A clear conclusion was required for full credit.*

3. Continued

By the product rule, $d(\mathrm{S_tE_t}) = \mathrm{E_t dS_t + S_t dE_t + dS_t dE_t}$.

Substituting yields, $d(\mathrm{S_tE_t}) = \mathrm{S_tE_t}\left((\mu + \rho\sigma_S\sigma_E)\mathrm{dt} + \sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}\,dZ_t\right)$.

We can apply Ito's Lemma to the above to get the desired SDE:
$d\left(ln(\mathrm{S_tE_t})\right) = \left(\mu - \frac{\sigma_S^2}{2} - \frac{\sigma_E^2}{2}\right)\mathrm{dt} + \sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}\,dZ_t$.

The final result does follow an arithmetic Brownian motion under measure $\mathbb{P}$.

Alternate approach:
$d\left(ln(\mathrm{S_tE_t})\right) = d(lnS_t) + d(lnE_t)$, where under Ito's Lemma

$d(lnS_t) = \left(\mu - \frac{\sigma_S^2}{2}\right)\mathrm{dt} + \sigma_S \mathrm{dW_t}$ and $d(lnE_t) = -\frac{\sigma_E^2}{2}\mathrm{dt} + \sigma_E \mathrm{dV_t}$

Thus, $d\left(ln(\mathrm{S_tE_t})\right) = \left(\mu - \frac{\sigma_S^2}{2} - \frac{\sigma_E^2}{2}\right)\mathrm{dt} + \sigma_S\mathrm{dW_t} + \sigma_E\mathrm{dV_t}$, which is an equivalent result.

(c) Show that $e^{-rt}S_tE_t$ is a martingale under the risk-neutral measure $\mathbb{Q}$ using Girsanov Theorem, with the numeraire being CAD risk-free assets.

Commentary on Question:
For full credit, candidates needed to show an understanding of Girsanov's Theorem as well as the form of a martingale under risk-neutral measure $\mathbb{Q}$. Many candidates only received partial credit for calculating the SDE of $e^{-rt}S_tE_t$.

S_tE_t by definition represents the Canadian asset price in CAD, making r the associated risk-free rate.

$$\begin{aligned} d(e^{-rt}S_tE_t) &= -re^{-rt}S_tE_tdt + e^{-rt}[E_tdS_t + dE_tS_t + dE_tdS_t,] \\ &= e^{-rt}S_tE_t\left((\mu - r + \rho\sigma_S\sigma_E)dt + \sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}\,dZ_t\right) \end{aligned}$$

By applying Girsanov's Theorem to change measure $\mathbb{P}$ to an equivalent risk-neutral measure $\mathbb{Q}$, we can utilize a process, $\tilde{\tilde{Z}}_t$, which is a standard Brownian motion under that measure. By letting,

$$d\tilde{\tilde{Z}}_t = dZ_t + \frac{(\mu - r + \rho\sigma_S\sigma_E)}{\sqrt{\sigma_S{}^2 + \sigma_E{}^2 + 2\rho\sigma_S\sigma_E}}$$

3. Continued

we find under $\mathbb{Q}$, $d(e^{-rt}S_tE_t) = e^{-rt}S_tE_t\left(\sqrt{{\sigma_S}^2 + {\sigma_E}^2 + 2\rho\sigma_S\sigma_E}\,d\tilde{\tilde{Z}}_t\right)$.

As there is no drift, $e^{-rt}S_tE_t$ is a $\mathbb{Q}$-martingale.

4. Learning Objectives:

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2c) Calibrate a model to observed prices of traded securities

(2b) Understand and apply various one-factor interest rate models.

(2f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors and swaptions.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014 (Ch. 3, 4, 8, 9, 10)

Problems and Solutions in Mathematical Finance Chin and Olafssson

Commentary on Question:

The candidates did well on this question.

Solution:

(a) Prove by using Ito's lemma that

$$\int_0^T f(t)g'(W_t)\,dW_t$$
$$= f(t)g(W_t)|_0^T - \int_0^T f'(t)g(W_t)dt - \frac{1}{2}\int_0^T f(t)g''(W_t)\,dt.$$

Commentary on Question:

This part is straightforward application of Ito's Lemma, candidates did well on it.

By Ito's Lemma:

$$dF_t = df(t)g(W_t) + f(t)dg(W_t)$$
$$= f'(t)g(W_t)dt + f(t)[g'(W_t)dW_t + \frac{1}{2}g''(W_t)dt]$$

Integrating both sides and rearranging the terms we get:

$$\int_0^T f(t)g'(W_t)\,dW_t$$
$$= [f(t)g(W_t)]|_0^T - \int_0^T f'(t)g(W_t)\,dt - \frac{1}{2}\int_0^T f(t)g''(W_t)\,dt$$

4. Continued

(b) Determine $Corr(I_T, Y_T)$.

Commentary on Question:
This part can be done by using part (a) or the definition of the correlation. Candidates did well on it.

We have:

$$Corr(I_T, Y_T) = \frac{Cov(I_T, Y_T)}{\sqrt{Var(I_T)Var(Y_T)}}$$
$$Corr(I_T, Y_T) = \frac{T^3/6}{\sqrt{\left(\frac{T^3}{3}\right)(\frac{T^3}{3})}}$$

Using part (a) with $F_t = t\,W_t$, we get:

$$I_T = TW_T - \int_0^T W_t dt \Rightarrow I_T + Y_T = TW_T$$

Therefore:

$$Cov(I_T + Y_T, I_T + Y_T) = Var(TW_T)$$
$$Var(I_T) + Var(Y_T) + 2Cov(I_T, Y_T) = T^2 Var(W_T)$$
$$\frac{T^3}{3} + \frac{T^3}{3} + 2Cov(I_T, Y_T) = T^3 \; therefore \; Cov(I_T, Y_T) = \frac{T^3}{6}$$

And $Corr(I_T, Y_T) = 1/2$
Another way:

$$\text{Cov}(I_T, Y_T) = E[I_T . Y_T] - E[I_T]E[Y_T] = E[I_T . Y_T], since\ E[I_T] = 0$$

$$\text{Cov}(I_T, Y_T) = E\left[\int_0^T t dW_t \int_0^T W_t dt\right] = E\left[\int_0^T t dW_t \left(TW_T - \int_0^T t dW_t\right)\right]$$
$$= E\left[\int_0^T t\,(TW_T - t) dt\right] = \frac{T^3}{2} - \frac{T^3}{3} = \frac{T^3}{6}$$

(c) Show that

$$\int_0^T \frac{2W_t}{1+W_t^2}\, dW_t = \ln(1 + W_T^2) - \int_0^T \frac{1-W_t^2}{\left(1+W_t^2\right)^2}\, dt.$$

4. Continued

Commentary on Question:
This part can be done using part (a). Most candidates did well on it.

Let $F_t = f(t)\, g(W_t) = \ln(1 + W_t^2)$. Then:

$$f'(t) = 0 \text{ and}$$

$$g'(x) = \frac{2W_t}{1 + W_t^2}$$
$$g''(x) = \frac{2(1 - W_t{}^2)}{\left(1 + W_t{}^2\right)^2}$$

From part (a):

$$\int_0^T g'(W_t)\, dW_t = [g(W_t)]|_0^T - \frac{1}{2}\int_0^T g''(W_t)\, dt$$

Substituting $\ln(1 + W_t^2)$ for g:

$$\int_0^T \frac{2W_t}{1+W_t^2}\, dW_t = ln(1 + W_T^2) - \int_0^T \frac{1-W_t^2}{\left(1+W_t^2\right)^2}\, dt$$

(d) Show that $\int_0^T \frac{1-W_t^2}{\left(1+W_t^2\right)^2} dt \leq T.$

Integrate both sides of:

$$\frac{1 - W_T^2}{(1 + W_T^2)^2} \leq 1$$

Then:

$$\int_0^T \frac{1 - W_t^2}{(1 + W_t^2)^2} dt \leq \int_0^T dt = T$$

4. Continued

(e) Show that $E[\ln(1 + W_T^2)] \leq T$.

Commentary on Question:
This part can be done using parts (c) and (d). Candidates did well on it

Take expected values from both sides of the results in part (c).

$$E[\int_0^T \frac{2W_t}{1+W_t^2}\, dW_t] = E[\ln(1 + W_T^2)] - E[\int_0^T \frac{1-W_t^2}{(1+W_t^2)^2}\, dt]$$

Since

$$E\left[\int_0^T \frac{2W_t}{1 + W_t^2}\, dW_t\right] = 0$$

And rearranging we get:

$$E[\ln(1 + W_T^2)] = E\left[\int_0^T \frac{1-W_t^2}{(1+W_t^2)^2}\, dt\right] \leq E[T] = \mathrm{T}$$

5. **Learning Objectives:**

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2b) Understand and apply various one-factor interest rate models.

(2d) Describe the practical issues related to calibration, including yield curve fitting.

Sources:

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Pietro, 2010; Chapter 19

QFIQ-136-23: Calibrating Interest Rate Models

Commentary on Question:

The question tested candidates on quantitative tools and techniques for modeling the term structure of interest rates with the Hull-White model. Candidates performed well on part (e) which tested their ability to calculate the price of an option. Candidates performed poorly on parts (a)-(d).

Solution:

(a) Explain whether the fitted model is a true arbitrage-free model.

Commentary on Question:

Most candidates answered this question incorrectly and stated that the Hull-White model is an arbitrage-free model.

The fitted yield curve is obtained by fitting a third degree polynomial to 20 points. It may have done through least square or "lm" using R.
It is very unlikely the fit will be perfect as least square fit just minimizes the errors, not making them zero.

Also gamma and sigma are obtained by fitting five cap prices so those estimates with the 3 degree polynomial would be unlikely to produce a perfect fit to yield curve and cap prices, so the calibrated model is not truly arbitrage free.

(b) Derive an expression for the instantaneous forward rate at time 0 $f(0,t)$.

Commentary on Question:

Candidates performed below expectation on this question. A common mistake was to take the derivative of r(0,t).

5. Continued

$$f(0,t) = \frac{\partial}{\partial t}\big(t\, r(0,t)\big)$$

$$f(0,t) = 0.01091858598 + \ 0.01251008594 * 2 * t \ - 0.000140114635 * 3 * t^2 + \ 0.005654825 * 4 * t^3$$

$$f(0,t) = \ 0.010918586 + \ 0.025020172t \ - 0.0004203439t^2 \ + 0.0226193t^3$$

(c) Derive an expression for θ_t .

Commentary on Question:
Many candidates were able to identify the correct initial formula to use. Most candidates struggled to convert the formula to decimal numbers.

$$\theta_T = \frac{\partial f(0,T)}{\partial T} + \gamma f(0,T) + \frac{\sigma^2}{2\gamma}(1 - exp(-2\gamma T))$$

As f(0,T) has been calculated as percentage points and σ^2 is in the equation, we need to convert f(0,T) and σ to deimal numbers before using the formula.

$$\theta_t = \sum_{i=1}^{n} a_i i(i+1)t^{i-1} + \sum_{i=0}^{n} \gamma a_i (i+1)t^i + \frac{\sigma^2}{2\gamma}(1 - exp(-2\gamma t))$$

$$\theta_t = \ 0.02709470321 + 0.0039131448472t + 0.06777803465805t^2 + \ 0.004297667t^3 + 0.001010947(1 - exp\,(-0.38t)$$

(d) Compute $E[r_{1.25}|r_1 = 0.03\%]$, given $f(0{,}1.25) = 0.036068$.

Commentary on Question:
Most candidates performed poorly on this question. To receive full credit, candidates needed to use the appropriate formula and calculate the expectation correctly. An alternative answer was accepted if candidates assumed r_1=3%.

$$E[r_{t+s}|r_t] = \ r_t exp(-\gamma s) + \ f(0,s+t) - \ f(0,t)exp(-\gamma s) + \frac{\sigma^2}{2\gamma^2}\left[1 \ - exp(-\gamma s) + exp\big(-2\gamma(t+s)\big) \ - \ exp\big(-\gamma(2t+s)\big)\right]$$

$E[r_{1.25}|r_1 = o.o3\%] = \ 0.000286 + 0.036068 - 0.055441$ +0.005321*0.016137

$E[r_{1.25}|r_1 = o.o3\%] = \ -1.9\%$

5. **Continued**

(e) Compute the price of a European call option on a zero-coupon bond with the following specifications:

- The underlying bond is a 4-year zero-coupon bond at issue of the option.
- The option matures in one year.
- The strike price is 80 out of 100 principal.

Commentary on Question:
Candidates performed well on this question. Most candidates were able to receive partial credit even if they did not get to the final answer. We accepted a range of answers depending on the inputs used.

Use Veronesi text formula 19.30-19.33

First calculate $Z(r_0, 0; T_O)$ and $Z(r_0, 0; T_B)$ using $Z(r_0, 0; T) = exp\,(-T * \frac{r(0,T)}{100})$ with $T_O = 1, T_B = 4.$

$$Z(r_0, 0; T_O) = exp\left(-\frac{r(0,1)}{100}\right) = exp\left(-\frac{2.8943}{100}\right) = 0.971471$$

$$Z(r_0, 0; T_B) = exp\left(-4 * \frac{r(0,4)}{100}\right) = exp\left(-4 * \frac{4.215968}{100}\right) = 0.8448141$$

Calculate

$$B(T_O; T_B) = B(0; T_B - T_0) = \frac{1}{\gamma}(1 - exp\big(-\gamma(T_B - T_O)\big) = 2.2867$$

Calculate

$$S_Z(T_O; T_B) = B(T_O; T_B)\sigma\sqrt{\frac{1 - exp(-2\gamma T_O)}{2\gamma}} = 0.04088$$

Calculate $d_1 = \frac{1}{S_Z(T_O;T_B)} log\left(\frac{Z(r_0,0;T_B)}{KZ(r_0,0;T_O)}\right) + \frac{S_Z(T_O;T_B)}{2} = 0.906958$

$d_2 = d_1 - S_Z(T_O; T_B) = 0.866078$

$$Call\ Price = Z(r_0, 0; T_B)N(d_1) - KZ(r_0, 0; T_O)\, N(d_2) = 0.063869$$

6. **Learning Objectives:**

2. The candidate will understand:
   - The Quantitative tools and techniques for modeling the term structure of interest rates.
   - The standard yield curve models.
   - The tools and techniques for managing interest rate risk.

Learning Outcomes:

(2b) Understand and apply various one-factor interest rate models.

(2f) Apply the models to price common interest sensitive instruments including: callable bonds, bond options, caps, floors, and swaptions.

(3k) Understand and apply multifactor interest rate models.

Sources:

An Introduction to the Mathematics of Financial Derivatives, Hirsa, Ali and Neftci, Salih N., 3rd Edition 2nd Printing, 2014

Fixed Income Securities: Valuation, Risk, and Risk Management, Veronesi, Piertro, 2010

Commentary on Question:

Candidates should be able to show their understanding of one-factor interest rate models. This involves using the model to assess the value of callable bonds and recognizing any limitations of the one-factor approach. Most candidates did not perform well in the application.

Solution:

(a) Show that for $s < t$

(i) $E[r_t|\mathcal{F}_s] = r_s e^{-\alpha(t-s)} + m\left(1 - e^{-\alpha(t-s)}\right)$,

(ii) $Var[r_t|\mathcal{F}_s] = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(t-s)}\right)$.

Commentary on Question:

The majority of candidates performed well, but a few failed to mention the quadratic variation when using Ito's formula.

Set $g(r_t, t) = e^{\alpha t} r_t$.

Then $dg = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial r}dr + \frac{\frac{1}{2}\partial^2 g}{\partial r^2} < dr, dr >= \alpha e^{\alpha t} r_t dt + e^{\alpha t} \cdot (\alpha(m - r_t)dt + \sigma dW_t) + 0 = \left(\alpha e^{\alpha t} r_t + \alpha e^{\alpha t}(m - r_t)\right)dt + e^{\alpha t}\sigma dW_t = \alpha e^{\alpha t}(m - r_t)dt + e^{\alpha t}\sigma dW_t$

6. Continued

$$g(r_\nu,\nu)|_{\nu=s}^{\nu=t} = e^{\alpha t} r_t - e^{\alpha s} r_s = \alpha \cdot m \int_s^t e^{\alpha \nu}\, d\nu + \sigma \int_s^t e^{\alpha \nu}\, dW_\nu$$
$$= m(e^{\alpha t} - e^{\alpha s}) + \sigma \int_s^t e^{\alpha \nu}\, dW_\nu$$
$$r_t = r_s e^{-\alpha(t-s)} + m\left(1 - e^{-\alpha(t-s)}\right) + \sigma e^{-\alpha t} \int_s^t e^{\alpha \nu}\, dW_\nu$$
$$E[r_t|\mathcal{F}_s] = E\left[r_s e^{-\alpha(t-s)} + m\left(1 - e^{-\alpha(t-s)}\right) + \sigma e^{-\alpha t} \int_s^t e^{\alpha \nu}\, dW_\nu \mid \mathcal{F}_s\right]$$
$$= r_s e^{-\alpha(t-s)} + m\left(1 - e^{-\alpha(t-s)}\right)$$
$$V[r_t|\mathcal{F}_s] = V\left[\sigma e^{-\alpha t} \int_s^t e^{\alpha \nu}\, dW_\nu |\, \mathcal{F}_s\right] = \sigma^2 e^{-2\alpha t} \int_s^t e^{2\alpha \nu} d\nu = \frac{\sigma^2}{2\alpha}\left[1 - e^{-2\alpha(t-s)}\right]$$

(b) Show that $B(t,T)$ follows a log-normal distribution.

Commentary on Question:
Although most candidates were unable to expand the stochastic integral, they did correctly identify the direction and were awarded partial credit.

$$\int_t^T r_u du = m(T-t) - (m - r_t) \int_t^T e^{-\alpha(u-t)} du$$
$$+ \sigma \int_t^T \int_t^u e_s^{-\alpha(u-s)} dW_s du$$

$$= m(T-t) - \frac{1}{\alpha}(m - r_t)\left(1 - e^{-\alpha(T-t)}\right)$$
$$+ \sigma \int_t^T \int_t^u e^{-\alpha(u-s)}\, dW_s du$$
$$\therefore E\left[-\int_t^T r_u du\right] = -m(T-t) + \frac{1}{\alpha}(m - r_t)\left(1 - e^{-\alpha(T-t)}\right)$$

Also from the Fubini's theorem in integral,

$$\sigma \int_t^T \int_t^u e^{-\alpha(u-s)}\, dW_s du = \sigma \int_t^T \int_s^T e^{-\alpha(u-s)} du\, dW_s = \frac{\sigma}{\alpha} \int_t^T \left(1 - e^{-\alpha(T-s)}\right) dW_s$$
$$\because\ t < s < u, \qquad t < u < T \rightarrow s < u < T, \qquad t < s < T$$

By Ito isometry,

$$V\left[-\int_t^T r_u du\right] = \frac{\sigma^2}{\alpha^2} \int_t^T \left(1 - e^{-\alpha(T-s)}\right)^2 ds$$
$$= \frac{\sigma^2}{\alpha^2}(T-t) - \frac{2\sigma^2}{\alpha^3}\left(1 - e^{-\alpha(T-t)}\right) + \frac{\sigma^2}{2\alpha^3}\left(1 - e^{-2\alpha(T-t)}\right)$$

6. Continued

Since both $E\left[-\int_t^T r_u du\right]$ and $V\left[-\int_t^T r_u du\right]$ are non-random, $-\int_t^T r_u du$ is normally distributed. Hence, $B(t,T)$ is log-normally distributed.

(c) Show that callable bond price at the call maturity is $min(\text{non callable Bond Price, Strike Price of the call})$ when callable bond is a combination of a straight bond and a call option on the bond.

Commentary on Question:
Many candidates made a mistake by assuming that a callable bond is the sum of a non-callable bond and a call option. Candidates did not receive any credit for this incorrect fundamental approach.

Callable bond is a combiantion of common bond and short of call option on the bond for the perspective of bond buyer. The issuer can call the bond with predetermined price $K$when the interest rate is lower to refinance the loan. At the expiration, the payoff will be:

$$\begin{aligned} Callable\ Bond &= non\ Callable\ Bond - Call\ Option = B - Max(B-K,0) \\ &= B - Max(B,K) + K = B + Min(-B,-K) + K \\ &= Min(0, B-K) + K = Min(B,K) \end{aligned}$$

(d) Calculate

(i) the bond option price at time 0.

(ii) the callable bond price using the embedded call option price and the straight bond price

Commentary on Question:
Many candidates struggled with the application of the interest rate model, making errors in their calculations resulting in incorrect final answers. However, partial credit was given for any correct calculations made.

i.
Because the interest rate follows one factor model, call price for coupon bond is calcualted from the formula:

$$Call = \sum_{i=1}^{n} c(i) \times \left(Z(r_0, 0; T_i) \times N\left(d_1(i)\right) - K_i \times Z(r_0, 0; T_O) \times N\left(d_2(i)\right)\right)$$

- i: coupon period after the expiration time of the option
- n: maturity of bond at which the principal and coupon paid
- c(i): coupon payment (+ principal when i=n)
- $Z(r_0, 0; T_i)$: Zero coupon bond price maturing T_i at time 0
- $K_i = Z(r_K^*, T_O; T_i)$ where r_K^* is defined in the question.
- $Z(r_K^*, T_O; T_i)$: Zero coupon bond price maturing T_i at time T_O

6. **Continued**

- $d_1(i) = \frac{1}{S_Z(T_O;T_i)} log\left(\frac{Z(r_0,0;T_i)}{K_i \times Z(r_0,0;T_O)}\right) + \frac{S_Z(T_O;T_i)}{2}$
- $S_Z(T_O;T_i) = B(T_O;T_i) \times \sqrt{\frac{\sigma^2}{2\alpha}(1-e^{-2\alpha T_O})}$
- $B(T_O;T_i) = \frac{1}{\alpha}\left(1-e^{-\alpha(T_i-T_O)}\right)$
- *N()*: Standard normal cdf

Applying parameters in the formula

T_i	$A(T_O\ ;\ T_i)$	$B(T_O\ ;\ T_i)$	K_i	$Z(0;\ T_i)$	$S_Z\ (T_O\ ;\ T_i)$	$d_1\ (i)$	$Call$
1.5	-0.0034	0.4461	0.9764	0.9570	0.0080	0.3067	0.0062
2	-0.0126	0.7996	0.9518	0.9371	0.0143	0.3112	0.0108

$c(1.5)\ =$	2.5
$c(2)\ =$	102.5

$$Call(1.5) = 2.5 \times 0.0062 = 0.015$$
$$Call(2) = 102.5 \times 0.0108 = 1.11$$
$$Call\ price = Call(1.5) + Call(2) = 1.125$$

ii.
Coupon bond price is obtained from

$$P_c(r,t;T) = coupon \times \sum_{i=1}^{n} Z(r,t;T_i) + 100 \cdot Z(r,t;T_n)$$
$$Z(r,t;T) = e^{A(t,T)-B(t,T)\times r}$$

$$B(t;T) = \frac{1-e^{-\alpha(T-t)}}{\alpha},$$
$$A(t;T) = \big(B(t;T)-(T-t)\big)\left(m-\frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2 \cdot B(t,T)^2}{4\alpha},$$

T_i	$A(0\ ;\ T_i)$	$B(0\ ;\ T_i)$	r_0	$Z(0;\ T_i)$	$c(i)$	dcf
0.5	-0.0034	0.4461	1.62%	0.9894	2.5	2.4736
1	-0.0126	0.7996	1.62%	0.9747	2.5	2.4368
1.5	-0.0265	1.0797	1.62%	0.9570	2.5	2.3925
2	-0.0439	1.3017	1.62%	0.9371	102.5	96.0478

Bond price $P_c(r,t;T)$ = sum of discounted cash flow (dcf) = $\sum Z(r,t;T) \cdot CF$ =103.35.

6. Continued

Call price = 1.125

Hence the callable coupon bond price = 103.35-1.125 = 102.225

(e)

(i) Describe a shortcoming of a one-factor Vasicek model.

(ii) Explain how two-factor Vasicek model can resolve it.

Commentary on Question:
Although many candidates did not provide complete answers, they exhibited a strong comprehension of the limitations of the one-factor Vasicek model.

We need at least three factors to explain the variation in yields. In other words, the yield curve not only moves up and down, but it also changes slope as well as curvature. Some interest rate models do not have features for independent variation of these quantities: For instance, in the Vasicek model, the level, slope, and curvature of the yield curve are all tied to the short-term interest rate r_t, and thus they are perfectly correlated.
The methodology extends to many factors, including the risk neutral pricing methodology.
For instance, the two factor Vasicek model is given by:

$$dr_t = \left[\gamma_1^*(\bar{\phi}_1^* - r_t) + \gamma_2^*\bar{\phi}_2 + (\gamma_1^* - \gamma_2^*)\phi_{2,t}\right]dt + \sigma_1 dX_1 + \sigma_2 dX_2$$

$$d\phi_{2,t} = \gamma_2^*\left(\bar{\phi}_2^* - \phi_{2,t}\right)dt + \sigma_2 dX_{2,t}$$

The short-term rate follows a mean reverting process, as in the standard Vasicek model, but now its risk neutral drift depends on the second factor $\phi_{2,t}$. Assuming $\gamma_1^* - \gamma_2^* > 0$, for instance, then for given current rate r_t, when the second factor $\phi_{2,t}$ increases, the risk neutral expectation of future short-term rate increases, which in turn implies a steepening of the term structure of interest rates. In other words, the factor $\phi_{2,t}$ affects the slope of the
term structure, in addition to its movement implied by r_t.
The long-term yield is expressed by a function of the short-term rate r_t plus another factor $\phi_{2,t}$.

$$r_{t(\tau)} = -\frac{A(\tau)}{\tau} + \frac{B_1(\tau)}{\tau} r_t + \frac{C(\tau)}{\tau}\phi_{2,t}$$

The two-factor model partly decouples the long-term yield $r_t(\tau)$ from the short-term rate r_t. Indeed, for given r_t we now may have different long-term yields $r_t(\tau)$, which depend on $\phi_{2,t}$.

7. **Learning Objectives:**

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(3h) Compare and contrast the various kinds of volatility, e.gl, actual, realized, implied and forward, etc.

Sources:

The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

QFIQ-115-17: Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios

Solution:

(a)

(i) Show by using Ito's Lemma that

$$dY_t = [\left(\gamma^*\overline{r^*} - \frac{1}{2}\alpha\right)e^{-Y_t} - \gamma^*]dt + \sqrt{\alpha}e^{-\frac{Y_t}{2}}dX_t$$

(ii) Explain why the drift term of dY_t is positive if Y_t gets too far below from 0.

Commentary on Question:

Candidates performed well on this question part. Most candidates were able to correctly recall and apply Ito's Lemma to derive the process for dY_t. Candidates needed to specify that the inequality $\gamma\,\bar{r} > \frac{1}{2}\alpha$ contributes to positive drift to receive full credit.

(i) By Ito's Lemma, $dY_t = \frac{1}{r}dr - \frac{1}{2r^2}(dr)^2$

$$= e^{-Y_t}\left(\gamma(\bar{r} - r_t)dt + \sqrt{\alpha r_t}dX_t\right) - \frac{1}{2}\frac{1}{r_t^2}\,\alpha r_t\,dt$$

Note: $dt^2 = dt\,dXt = 0, dX_t^2 = dt,\ r_t = e^{Y_t}, \sqrt{r_t} = e^{\frac{Y_t}{2}}$

7. Continued

$$=e^{-Y_t}\left(\gamma(\bar{r}-r_t)dt+\sqrt{\alpha r_t}dX_t\right)-\frac{1}{2}\frac{1}{r_t}\ \alpha\ dt$$

$$=e^{-Y_t}\left(\gamma(\bar{r}-e^{Y_t})dt+\sqrt{\alpha}e^{\frac{Y_t}{2}}dX_t\right)-\frac{1}{2}e^{-Y_t}\ \alpha\ dt$$

$$=\left(\gamma\bar{r}-\frac{1}{2}\alpha\right)e^{-Y_t}-\gamma]dt+\sqrt{\alpha}e^{-\frac{Y_t}{2}}dX_t$$

(ii) As $\gamma\,\bar{r}>\frac{1}{2}\alpha$, if Y_t gets too far below from 0, the drift term of dY_t will become strongly positive as e^{-Y_t} will be very large.

(b)

(i) Prove that $\lim_{t\to\infty} E[r_t\,|r_0]=\bar{r}^*$ and $\lim_{t\to\infty} Var[r_t\,|r_0]=\frac{\bar{r}^*\alpha}{2\sigma}$

(ii) Identify the distribution of $ln\,Z(r_t,t;\,T)$.

Commentary on Question:

Few candidates received full credit on this question part. While many candidates could correctly prove the expectation and variance in part (i), few candidates correctly identified the correct distribution of $ln\,Z(r_t,t;\,T)$

(i) $\lim_{t\to\infty} \boldsymbol{E}[\boldsymbol{r_t}\,|\boldsymbol{r_0}]=\lim_{t\to\infty}\bar{\boldsymbol{r}}+(\boldsymbol{r_0}-\bar{\boldsymbol{r}})\boldsymbol{e^{-\gamma t}}=\ \bar{\boldsymbol{r}}+\boldsymbol{0}=\ \bar{\boldsymbol{r}}$

$$\lim_{t\to\infty}\boldsymbol{Var}[\boldsymbol{r_t}\,|\boldsymbol{r_0}]=\lim_{t\to\infty}[\frac{\boldsymbol{\alpha}}{\boldsymbol{\gamma}}\ \mathbf{r_0}\boldsymbol{e^{-\gamma t}}(\mathbf{1}-\boldsymbol{e^{-\gamma t}})+\frac{\boldsymbol{\alpha}\bar{\mathbf{r}}}{\mathbf{2}\boldsymbol{\gamma}}(\mathbf{1}-\boldsymbol{e^{-\gamma t}})^{\mathbf{2}}]$$

$$=\ \mathbf{0}+\frac{\boldsymbol{\alpha}\bar{\mathbf{r}}}{\mathbf{2}\boldsymbol{\gamma}}+\mathbf{0}=\frac{\boldsymbol{\alpha}\bar{\mathbf{r}}}{\mathbf{2}\boldsymbol{\gamma}}$$

Note: Some candidates invoked the fact that as a CIR model is mean-reverting, $\lim_{t\to\infty}\boldsymbol{E}[\boldsymbol{r_t}\,|\boldsymbol{r_0}]=\bar{\boldsymbol{r}}^*$. Full credit was given for this approach.

(ii) $\boldsymbol{Z}(\boldsymbol{r_t},\boldsymbol{t};\boldsymbol{T})=\boldsymbol{e^{A(t;T)-B(t;T)r_t}}$.

$\boldsymbol{lnZ}(\boldsymbol{r_t},\boldsymbol{t};\boldsymbol{T})=\ \boldsymbol{A}(\boldsymbol{t};\boldsymbol{T})-\boldsymbol{B}(\boldsymbol{t};\boldsymbol{T})\boldsymbol{r_t}$

This is a linear transformation of non-central chisqure distributions. As $\boldsymbol{r_t}$ is noncentral chi-square distributed, $\boldsymbol{lnZ}(\boldsymbol{r_t},\boldsymbol{t};\boldsymbol{T})$ is noncentral chi-square distributed.

7. Continued

(c)

(i) Calculate γ^*, α.

(ii) Calculate $A(0;10)$, $Z(r_0, 0; 10)$.

Commentary on Question:
While many candidates correctly approached this question part, few correctly calculated the requested quantities without errors. This question part did not require candidates to have solutions to parts (a) or (b).

(i) $\boldsymbol{B(t;T) = \frac{2\,(e^{(\psi_1(T-t)}-1)}{(\gamma^*+\psi_{1)}(\,e^{(\psi_1(T-t)}-1)+2\,\psi_1}}$

$$\boldsymbol{\gamma^* = \frac{\left[\frac{2\,\left(e^{(\psi_1(T-t)}-1\right)}{B(t;T)} - 2\,\psi_1\right]}{e^{(\psi_1(T-t)}-1} - \psi_1}$$

$$\boldsymbol{T = 10, t-0, B(0;10) = 4.256073,\ \ \psi_1 = \sqrt{\gamma^{*2}+2\alpha} = 0.212132}$$

$$\boldsymbol{\gamma^* = \frac{\left[\frac{2\,\left(e^{(0.212132(10)}-1\right)}{4.256073} - 2 * 0.212132\right]}{e^{(0.212132(10)}-1} - 0.212132}$$

$$\boldsymbol{\gamma^* = 0.2}$$

$$\boldsymbol{\psi_1 = \sqrt{\gamma^{*2}+2\alpha}}$$

$$\boldsymbol{\psi_1^2 = \gamma^{*2}+2\alpha}$$

$$\boldsymbol{\alpha = \frac{\psi_1^2-\gamma^{*2}}{2} = (0.212132^2 - 0.2^2)/2 = 0.0025}$$

(ii) $\boldsymbol{A(t;T) = \frac{2(\bar{r}^*\gamma^*)}{\alpha}\ln\left(\frac{2\,\psi_1\left(e^{(\psi_1+(\gamma^*)(T-t)/2}-1\right)}{(\gamma^*+\psi_{1)}(\,e^{(\psi_1(T-t)}-1)+2\,\psi_1}\right)}$

$$\boldsymbol{A(0;10) = 2 * 0.041 * \frac{0.2}{0.0025} * \ln\left(\frac{3.330961}{3.450197}\right)}$$
$$\boldsymbol{= -0.23071785}$$

$$\boldsymbol{Z(r_t, t; T) = e^{A(t,T)-B(t,T)r_t}}$$

$$Z(0.04, 0; 10) = \exp(= -0.23071785 - 4.256073 * 0.04) = 0.669676$$

8. Learning Objectives:

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3c) Demonstrate an understating of the different approaches to hedging – static and dynamic.

(3d) Demonstrate an understanding of how to delta hedge, and the interplay between hedging assumptions and hedging outcomes.

(3e) Analyze the Greeks of common option strategies.

Sources:

QFIQ-120-19: Chapters 6 and 7 of Pricing and Hedging Financial Derivatives, Marroni, Leonardo and Perdomo, Irene, 2014

QFIQ-115-17: Which Free Lunch Would You Like Today, Sir?: Delta Hedging, Volatility Arbitrage and Optimal Portfolios

Commentary on Question:

The question assessed candidates' understanding of option pricing, Greeks, and the limitations of model simplifications. The majority of candidates performed well in parts a and c. However, some candidates relied solely on a single formula to solve the answers, making it challenging for graders to award partial credit if the answers were incorrect. Additionally, most candidates approached part d from only one perspective, neglecting comments from both directional and non-directional traders. It is crucial to clearly indicate the correctness of statements and provide explanations. Many candidates overlooked the limitations of the Black-Scholes formula and the presence of transaction costs in real-world scenarios.

8. Continued

Solution:

(a) Draw the two Gamma curves in the Excel spreadsheet.

Commentary on Question:
It is beneficial for candidates to break down formulas into individual steps as it allows graders to award partial credit even if the final answers are not entirely correct.

formula for the gamma of a plain vanilla call or put option is equal to:

$$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

See detail calculations in excel

(b) Determine which curve corresponds to which time-to-maturity. Justify your answer.

Commentary on Question:
Although the majority of candidates answered the question correctly, a few candidates failed to provide an explanation of the relationship between Vega and time to maturity, leading to the awarding of partial credits.

When the option expiry is far away (1-year or longer), the Vega shows "fat tail". As expiry approaches, the tail gets considerably thinner because Vega shrinks considerably and converges to zero for deep out-of-the-money options (for this put option) and deep in-the-money options.

Thus the line above (with squares) is the Vega curve of a long-dated expiry (12-month) and the line below (with triangles) is the Vega curve of a short-dated expiry (3-month)

(c) Draw the two Vega curves in the Excel spreadsheet.

Commentary on Question:
It is beneficial for candidates to break down formulas into individual steps as it allows graders to award partial credit even if the final answers are not entirely correct.

8. Continued

$$\text{Vega} = S\sqrt{T-t}N'(d_1)$$

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

Calculate the Vega under different expiries (0.25 vs 1.0) by plugging in other assumptions (stock, strike, interest rate, volatility). See detail calculations in excel

(d) Critique the following two claims from the perspectives of directional traders and non-directional traders, respectively.

(i) "When buy a put option, you cannot lose more than the option premium paid for it."

(ii) "To make a profit from the put purchase, realized volatility needs to be higher than the implied volatility."

Commentary on Question:

Most candidates approached part d from only one perspective, neglecting comments from both directional and non-directional traders. It is crucial to clearly indicate the correctness of statements and provide explanations.

(i) From the point view of directional traders, this is true. Directional traders buy the put option and do nothing else. If the put option expires out of money, they only lose the premium they paid for, nothing else.

However, from the point view of non-directional traders, this is not necessarily true, because non-directional traders usually delta-hedge their puts by purchasing underlying stocks. They may end up losing more on the delta hedging than the option payoff. This can happen when the stock declines slowly towards the strike but does not cross it. In this case, the trader loses the option premium when the put expires worthless and, in addition, incurs losses on the underlying stock purchased for delta-hedging.

(ii) From the point view of directional traders, this is true.

However, from the point view of non-directional traders, the profit is not guaranteed even if the realized volatility is higher than the implied volatility because the non-directional traders need to frequently rebalance their delta-hedged portfolio.

8. **Continued**

The rebalancing cannot be continuously executed in practice as assumed in the Black-Scholes model and the delta-hedging is highly path dependent and the different timing of rebalancing can lead to different outcomes.

9. Learning Objectives:

3. The candidate will understand:
   - How to apply the standard models for pricing financial derivatives.
   - The implications for option pricing when markets do not satisfy the common assumptions used in option pricing theory.
   - How to evaluate risk exposures and the issues in hedging them.

Learning Outcomes:

(3a) Demonstrate an understanding of option pricing techniques and theory for equity derivatives.

(3b) Identify limitations of the Black-Scholes-Merton pricing formula.

(3k) Describe and contrast several approaches for modeling smiles, including: stochastic volatility, local-volatility, jump-diffusions, variance-gamma, and mixture models.

(3l) Explain various issues and approaches for fitting a volatility surface.

Sources:

CH.14, CH.17 The Volatility Smile, Derman, Emanuel and Miller, Michael B., 2016

Commentary on Question:

It was discovered there was a typo in the question regarding the formula for volatility. The following formula was provided in the exam that caps σ at 2%:

$$\sigma = \min(15\% - 1.5 * \frac{S - S_0}{S_0}, 2\%)$$

The intended formula is to floor σ at 2%:

$$\sigma = \max(15\% - 1.5 * \frac{S - S_0}{S_0}, 2\%)$$

Candidates received credits for correctly applying either formula. In the solutions below, the results using both formulas are shown.

Solution:

(a) Construct the first four levels of a binomial tree for volatility and stock price with $\Delta t = \frac{1}{52}$ years by using the Cox-Ross-Rubinstein approach to construct the central spine of the tree. Show calculation for the first two levels clearly.

Commentary on Question:

Candidates received partial credit for writing down the binomial tree formula. Candidates generally demonstrated the ability for computing the first level of each tree (i.e. volatility, stock price).

9. **Continued**

For the 2nd level, most candidates failed to compute the central spine accurately by simply writing down S_0 instead of using the forward stock price formula. Not many candidates attempted to calculate binomial tree beyond level 2.

The inputs used for the formula are as below:

$d_t = 0.019; S_0 = 105; r = 0.04;\ b = 0.01; K = 102$

<u>*Based on the formula that floors σ at 2%:*</u>

$$\sigma_0 = \max(15\% - 1.5 * \frac{105 - 105}{105}, 2\%) = 0.15$$

Applying the following formulas for level 1 stock price:

$$S_u = Se^{\sigma(S,t)\sqrt{dt}}$$
$$S_d = Se^{-\sigma(S,t)\sqrt{dt}}$$

$S_u = 105 *\ e^{0.15*\sqrt{\frac{1}{52}}} = 107.21$ and $S_d = 105 *\ e^{-0.15*\sqrt{\frac{1}{52}}} = 102.84$

$$\sigma_u = \max(15\% - 1.5 * \frac{107.21 - 105}{105}, 2\%) = 0.118$$

Apply the following formulas for level 2+ stock prices:

$$S_u = F + \frac{S^2\sigma^2(S,t)dt}{F - S_d}$$
$$S_d = F - \frac{S^2\sigma^2(S,t)dt}{S_u - F}$$

where $F = Se^{(r-b)dt}$

$$S_{ud} = 105 *\ e^{(0.04-0.01)*2*\sqrt{\frac{1}{52}}} = 105.12$$

$$S_{uu} = 107.21 *\ e^{(0.04-0.01)*\sqrt{\frac{1}{52}}} + \frac{107.21^2 *\ 0.118^2 * \frac{1}{52}}{107.21 *\ e^{(0.04-0.01)*\sqrt{\frac{1}{52}}} - 105.12}$$

$$\sigma_u = \max(15\% - 1.5 * \frac{102.84 - 105}{105}, 2\%) = 0.181$$

9. Continued

$$S_{dd} = 102.84 * e^{(0.04-0.01)*\sqrt{\frac{1}{52}}} - \frac{102.84^2 * 0.181^2 * \frac{1}{52}}{105.12 - 102.84 * e^{(0.04-0.01)*\sqrt{\frac{1}{52}}}} = 99.91$$

Four levels of the volatility and pricing trees are shown as below:

Volatility tree

			7.5%	
		9.7%		
	11.8%		11.7%	
15.0%		14.8%		
	18.1%		17.9%	
		22.3%		
			26.7%	

Pricing tree

				111.18
			110.23	
		108.71		108.80
	107.21		107.31	
105.00		105.12		105.24
	102.84		102.98	
		99.91		100.08
			96.81	
				92.87

Based on the formula that caps σ at 2% as it appears on the exam:

$$\sigma = \min(15\% - 1.5 * \frac{S - S_0}{S_0}, 2\%)$$

The four levels of volatility tree and pricing tree can be calculated as below:

9. Continued

Volatility tree

			2.0%	
		2.0%		
	2.0%		2.0%	
2.0%		2.0%		
	2.0%		2.0%	
		2.0%		
			2.0%	

Pricing tree

				106.45
			106.02	
		105.72		105.84
	105.29		105.41	
105.00		105.12		105.24
	104.71		104.83	
		104.53		104.65
			104.24	
				104.06

(b) Calculate the price of a European put option with strike $K = 102$ that expires after four time-steps.

Commentary on Question:
Majority of candidates did not attempt this part of the question. To receive full credit, candidates needed to calculate the option payoff and their respective risk neutral probabilities and discount rates to arrive at the option price. Partial credit was given is to candidates if they could describe how to determine the price using the 3 components above.

Compute the payoff using the following formula:

$$q = \frac{F - S_d}{S_u - S_d}$$

<u>Based on the formula that floors σ at 2%:</u>

The risk-neutral probabilities can be calculated as below.

9. Continued

Risk-neutral transition prob.				
			62.82%	
		50.32%		
	59.79%		59.79%	
50.87%		50.89%		
	57.37%		57.37%	
		51.10%		
			55.42%	
RN Cumulative prob				
				9.61%
			15.30%	
		30.41%		29.52%
	50.87%		39.86%	
100.00%		48.64%		35.88%
	49.13%		34.59%	
		20.94%		20.42%
			10.24%	
				4.57%
Payoff				
0.8069				-
				-
				-
				1.92
				9.13

The option value is calculated as:

$$(1.92 * 20.42\% + 9.13 * 4.57\%)\ e^{-0.04*4*1/52} = 0.8069$$

Based on the formula that caps σ at 2% as it appears on the exam:

9. Continued

Risk-neutral transition prob.				
			38.58%	
		61.31%		
	38.52%		38.52%	
60.33%		60.33%		
	40.55%		40.55%	
		59.34%		
			40.50%	

RN Cumulative prob				
				5.50%
			14.25%	
		23.24%		24.58%
	60.33%		41.08%	
100.00%		53.18%		39.48%
	39.67%		35.08%	
		23.58%		24.74%
			9.59%	
				5.70%

Payoff				
-				-
				-
				-
				-
				-

The option value is zero.

(c) Calculate the Black-Scholes put option value.

Commentary on Question:
Many candidates received full credits. Those who calculated d_1 and d_2 correctly also received credit.

9. Continued

Apply the Black-Scholes formula,

$$d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} = \frac{\ln\frac{105}{102} + \left(0.04 - 0.01 + \frac{0.171^2}{2}\right)\frac{4}{52}}{0.171 * \sqrt{\frac{4}{52}}} = 0.6820$$

$$d_2 = d_1 - \sigma\sqrt{\tau} = 0.6344$$

$$p = Ke^{-r\tau}N(-d_2) - Se^{-b\tau}N(-d_1) = 0.753$$

(d) Explain why the Black-Scholes option value is different from the option value calculated using the binominal tree.

Commentary on Question:
Most candidates were able to identify at least one of the concepts below. For those that used the local volatility of 2% in part a., credit was given if they mentioned the difference between the 2% and the implied volatility of 17.1% in part c.

- With a constant volatility produces an option value converges to the BSM formula in the limit as the spacing between tree levels approaches zero. The time step here may not be small enough.
- The parameter of the local volatility model is not stationary and periodic recalibration is needed.
- Black-Scholes assumed constant volatility. However, in the local volatility model it is a function of the stock price.

10. Learning Objectives:

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4a) Identify and evaluate embedded options in liabilities, specifically indexed annuity and variable annuity guarantee riders (GMAB, GMDB, GMWB and GMIB).

(4b) Demonstrate an understanding of embedded guarantee risk including: market, insurance, policyholder behavior, and basis risk.

(4d) Demonstrate an understanding of target volatility funds and their effect on guarantee cost and risk control.

Sources:

QFIQ-134-22: An Introduction to Computational Risk Management of Equity-Linked Insurance, Feng, 2018 (sections 1.2-1.3, 4.7,4.8 (background), 6.2-6.3)

QFIQ-124-20: Variable Annuity Volatility Management: An Era of Risk-Control

Commentary on Question:

The question is mainly trying to test the candidates understanding of the principles of volatility management strategies and ability to apply them when designing and managing a product with equity guarantee.

Solution:

(a) Describe the principal objectives for an insurer in designing an equity-based guarantee.

Commentary on Question:

Most candidates could list out the principal objectives for an insurer in designing an equity-based guarantee, but failed to demonstrate their understanding of these objectives with descriptions, especially for stabilizing ALM and hedging performance.

- Write profitable business:
  Do the volatility management strategies reduce the hedge cost (risk-neutral value) of the guaranteed?

- Stabilize ALM and hedging performance
  Do the volatility management strategies improve the key hedge ratio, in particular Vega?
  How well do volatility management strategies minimize hedge P&L losses during crisis?
  Can our risk management and hedge program effectively mirror the changing fund position? (i.e. less basis risk)

10. **Continued**

- Optimize capital requirement
  Do the volatility management strategies reduce Statutory reserve requirement (and volatility of reserve)?

(b) Calculate the guarantee cost at the end of year 1 (t=1) for the GMMB rider under each of the 3 volatility management strategies. (Initial deposit = $100)

Commentary on Question:
For Asset Transfer Program, some candidates were able to determine the percentage of portfolio that needed to be allocated in cash, given the volatility level. However, many failed to rebalance the portfolio based on the portfolio value at t=1.

For Capped volatility fund, many candidates knew that the portfolio remained 100% in equity as the level of volatility was still within the threshold at 60%.

For VIX-Indexed Fee, some candidates calculated the rider fee in bps correctly, but were unable to get to the correct dollar amount. These candidates failed to realize that the rider fee was charged at the beginning of the year (as stated in the question), and thus fee only incurred at t=1.

Very few candidates attempted to calculate the guaranteed cost by taking weighted averaged of the guaranteed payoff under the risk neutral probabilities and calculating the present value at t=1.

<u>Asset Transfer Program</u>
Rebalance at t = 1:
Guaranteed Ratio (G%) = 1- 81.87/100 = 18.13%
Allocation in equity (S) = 1 – G% = 81.87%
Allocation in cash = 18.13%
Equity: 0.8187 unit of Equity S ($67.03 =0.8187 * $81.87)
Cash: $14.84 (sold 0.1813 unit of Equity S = 0.1813 * $81.87)

Payoff at t = 2:
Node 2, u:
The investment value
= 0.8187 unit of equity S + cash = $149.18 * 81.87% + $14.87= $136.98
GMMB payoff = Max (100-136.98, 0) = 0
Node 2, d:
The investment value
= 0.8187 unit of equity S + cash = $44.93 * 81.87% + $14.87 = $51.62
GMMB payoff = Max (100-51.62, 0) = $48.37

10. Continued

Guaranteed cost at the end of year 1 (t=1)
= NPV (Guaranteed Payoff) - Rider Fee
= (0 * 37.02% + 48.37 * 62.98%) * exp(-2%) - 0 = $29.86

- Capped Volatility Fund (σ_{capped} = 60%)
- Rebalance at t = 1:
- Allocation in equity (S) = σ_{capped}/ σ_{s1} = 60%/60% = 100%
- Allocation in cash = 0 %
- Equity: 1 unit of Equity S ($81.87 =1 * $81.87)

Payoff at t = 2:
Node 2, u:
The investment value
= 1 unit of equity S + 0 cash = $149.18
GMMB payoff = Max (100-149.18, 0) = 0
Node 2, d:
The investment value
= 1 unit of equity S + cash = $44.93
GMMB payoff = Max (100-44.93, 0) = $55.07

Guaranteed cost at the end of year 1 (t=1)
= NPV (Guaranteed Payoff) - Rider Fee
= (0 * 37.02% + 55.07 * 62.98%) * exp(-2%) - 0 = $33.99

<u>VIX- Index Fee</u>
Fee charged at t = 1:
Rider Fee = Max [0 bps, 200bps *(60% - 20%)] = 80 bps
Investment value *80bps = $81.87 * 80bps = 0.655

Rebalance at t = 1:
Sold 0.008 unit of equity S for the charged rider fee.
1- [(81.87 – 0.655)/ 81.87] = 1- 0.992 = 0.008

Payoff at t = 2:
Node 2, u:
The investment value
= 0.992 unit of equity S = $149.18 * 0.992 = 147.99
GMMB payoff = Max (100-147.99, 0) = 0
Node 2, d:
The investment value
= 0.992 unit of equity S = $44.93 * 0.992
GMMB payoff = Max (100-44.57, 0) = $55.43

10. **Continued**

Guaranteed cost at the end of year 1 (t=1)
= (0 * 37.02% + 55.43 * 62.98%) * exp(-2%) – 0.655
= 34.22 - 0.655 = $33.56

(c) Identify the 4 volatility management strategies from the table above including no volatility management strategy.

Commentary on Question:
Many candidates were able to identify strategy C and strategy B to be no volatility strategy and Asset Transfer Program respectively, but failed to provide justifications.

Lots of candidates failed to differentiate between D and E by recognizing that Capped Volatility could create protection against "tail spike" in volatility, and thus more effectively reducing the hedge P&L than VIX- Index Fee.

Strategy C: no volatility management strategy (or leverage on volatility).

- Higher guaranteed cost and hedge loss than the strategy of 100% static allocation in equity,

Strategy B: Asset Transfer Program.

- Highest reduction on guaranteed cost and hedge loss than the other strategies
    - Actively reallocate the fund (from equity to cash) when the portfolio becomes in-the-money at the defined trigger level.
    - More active risk-control than capped volatility and VIX-Indexed fee strategies.
    - As volatility spikes and equity value falls, the strategy is heavily invested in cash, leading the volatility level to be near expectation and stabilizing cash flow despite market fluctuation

Strategy D: Capped Volatility Fund Strategy

- Mild reduction on guaranteed cost (vs. 100% static allocation in equity)
    - Only activate when the equity volatility exceeds the cap level.
    - Given the cap at 60% (vs. the current at 20%), the strategy is expected to eliminate only a small portion of volatility cost.
- Lower hedge loss between D and E.
    - The volatility cap creates a protection against the "tail spike" in volatility, which can reduce the frequency and severity of the ultra-large returns, mitigating the hedge breakage

10. **Continued**

Strategy E: VIX-Indexed Fee Strategy

- Mild reduction on guaranteed cost and hedge loss (vs. 100% static allocation in equity)
  - The allocation in equity remains at 100%
  - The rider fee increases with the level of volatility, providing some offsets to guaranteed cost and hedge loss; however, given the fee level, the magnitude is expected to be small.
  - No protection against the "tail spike" in volatility and thus less effective than Asset Transfer Program or Capped Volatility in reducing the hedge loss.

(d)

(i) Calculate the Vega under each of the 3 volatility management strategies (Hint: use finite difference approximation).

(ii) Explain how low Vega can benefit the hedge program.

(iii) Propose a volatility management strategy from the insurer's perspective based on the results in part (c) and (d) (i).

Commentary on Question:
For part i), many candidates could correctly calculate the Vega given the provided data.

For part ii), most candidates knew that Vega was the sensitivity to the change in volatility, but failed to demonstrate their understanding on how low Vega could benefit the hedge program.

For part iii), only some candidates recognized that Asset Transfer Program is the strategy that best addresses the insurer's principal objectives in manufacturing an equity-based guarantee.

(i)
<u>Asset Transfer Program:</u>
Guarantee Cost (S0=100, $\sigma_{s,0}$= 10%) = 4.38
Guarantee Cost (S0=100, $\sigma_{s,0}$= 40%) = 10.35
Vega = (10.35 – 4.38)/(40%-10%) = 20.23

<u>Capped Volatility Fund:</u>
Guarantee Cost (S0=100, $\sigma_{s,0}$= 10%) = 7.55
Guarantee Cost (S0=100, $\sigma_{s,0}$= 40%) = 20.91
Vega = (20.91 – 7.55)/(40%-10%) = 44.53

10. Continued

VIX-Indexed Fee:
Guarantee Cost ($S0=100$, $\sigma s,0= 10\%$) = 7.35
Guarantee Cost ($S0=100$, $\sigma s,0= 40\%$) = 21.5
Vega = (21.5 – 7.35)/(40%-10%) = 47.17

(ii)
Vega is the rate of change in value of the portfolio with respect to the volatility of the underlying asset. Low Vega can stabilize the performance of hedge program.

(iii)
Given the result in d) -i) and c), Asset Transfer Program has the lowest guaranteed cost, the lowest hedge loss, and the smallest Vega when the volatility increases from 10% to 40%.
→ Best fit the objective of writing profitable business, as well as stabilizing ALM and hedging performance.

(e) Critique whether Joe's proposal meets the needs of the clients in the target market.

Commentary on Question:
Most candidates could recognize that Asset Transfer Program had the lowest equity allocation over time but failed to assess Joe's proposal based on the other two metrics.

The target clients value the upside investment potential and are willing to pay extra fees for it.

The three metrics used to measure the upside investment potential are:

i) Return and volatility profile
- Higher return relative to realized volatility is preferred
- Volatility management strategies do not alter the overall investment proposition much from a static 100% equity allocation strategy

ii) Equity allocation over time
- Higher allocation in equity has better "upside investment potential"

iii) Cumulative fee paid
- Additional fee paid for the volatility management strategy could reduce account value accumulation or decrease the guaranteed value.

Asset Transfer Program is the most active risk-control strategy among the three, rebalancing with cash based on the in-the-moneyness of the fund.

10. Continued

For i):

- o Asset Transfer Program is expected to have the return and volatility profile changed the most from a static 100% equity allocation fund.
- o VIX-Indexed Fee and Capped volatility fund likely offer a more similar return and volatility profile as a static 100% equity allocation fund. (The former has 100% allocation in equity, and for the later, rebalancing is only activated when equity volatility exceeds the cap at 60%.)

For ii):

- o Asset Transfer Program is expected to have the lowest equity allocation over time, due to the active risk control.
- o VIX-Indexed Fee is expected to have the highest equity allocation over time.

For iii):

- o VIX-Indexed Fee is the only strategy that would incur extra rider fee.
- o Given the result in b) (volatility spikes up to 60%), the extra rider fee doesn't have material impact to the account value accumulation over the rider term (2 years).
- o The target clients are willing to pay extra fees for the upside potential. Therefore, the fee saving of Asset Transfer Program over VIX-Indexed Fee may not add much value to the target clients.

➔ Asset Transfer Program doesn't meet the client's need. VIX-Indexed Fee better fits the need of target clients.

11. Learning Objectives:

4. The candidate will learn how to apply the techniques of quantitative finance to applied business contexts.

Learning Outcomes:

(4d) Demonstrate an understanding of target volatility funds and their effect on guarantee cost and risk control.

Sources:

QFIQ-124-20: Variable Annuity Volatility Management: An Era of Risk-Control

QFIQ-128-20: Mitigating Interest Rate Risk in Variable Annuities: An Analysis of Hedging Effectiveness under Model Risk

Commentary on Question:

The majority of candidates performed poorly on this question. Many candidates either did not attempt the question, or only attempted a limited part of the question.

Solution:

(a) Explain the considerations when using each of the approaches above.

Commentary on Question:

Most candidates answered this part adequately. Candidates were generally able to explain the differences and considerations between the three different approaches for calibrating the instantaneous variance process. No credit was awarded for only providing definitions.

(i) Since VAs have long term maturities, extracting appropriate implied volatilities will often involve unsound extrapolation.
Using implied volatilities relates to the fact that two models that are well calibrated to the implied volatility vanilla option surface may lead to very different prices and hedge ratios for exotic options.

(ii) The VIX index is constructed in a model free way, i.e. does not rely on the B-S model, therefore does not suffer from model risk. However, VIX is generally an upward biased forecast.

(iii) Historical volatility yields stable estimates over time. However would not reflect any forward-looking market expectations.

11. Continued

(b) Show that the insurer's expected present value of prospective rider fees becomes:

$$Y_t = \alpha G \left(\frac{1 - e^{-r(T-t)}}{r} \right) {}_{T-t}p_{x+t}$$

Commentary on Question:
Candidates' performances on this part varied greatly. Candidates needed to show steps to their derivation to receive credit. Candidates that did attempt the question performed well, but many candidates either did not attempt or did not show steps to their derivation.

Derive the value of prospective fees from first-principles:

$$L_t = \left(\int_t^T e^{-r(s-t)} \alpha G ds \right) {}_{T-t}p_{x+t}$$
$$= \alpha G \left(\frac{1 - e^{-r(T-t)}}{r} \right) {}_{T-t}p_{x+t}$$

(c) Explain whether the following has increased, decreased, or remained the same after this change, from the insurer's perspective.

(i) Delta of the liability net of rider fees.

(ii) Vega of the liability net of rider fees.

Commentary on Question:
Candidates performed poorly on this part. Many candidates did not correctly understand the impact of the change in rider fee to the delta and vega of the VA net liability. Credit was not awarded for just providing the answer without any rationale.

(i) Net liability delta has increased. Previous rider fee delta was positive and so it contributes to the negative delta from the GMMB put option. New rider fee no longer a function of the account value, so its delta is 0.

(ii) Net liability vega has not changed. Previous rider fee was 0. New rider fee is also not a function of volatility, so its vega is 0.

11. **Continued**

(d) Show that the fair value of prospective fees at time t, as defined as the risk-neutral expected present value of fees that will be collected by the insurer before the contract's maturity at time T, is:

$$L_t = G\left[(m+\lambda\theta)\left(\frac{1-e^{-r(T-t)}}{r}\right)+\lambda(v_t-\theta)\left(\frac{1-e^{-(r+\kappa)(T-t)}}{r+\kappa}\right)\right]{}_{T-t}p_{x+t}$$

Commentary on Question:
Candidates performed very poorly on this part, with many candidates skipping this question. Candidates needed to show steps to their derivation to receive credit. For those that did attempt the question, many either did not correctly integrate the stochastic variance term or did not provide any steps in their derivation.

Derive the value of prospective fees from first-principles:

$$\begin{aligned}
L_t &= \int_t^T e^{-r(s-t)}(m+\lambda E^{\mathbb{Q}}[v_s])G\,ds\ {}_{T-t}p_{x+t} \\
&= \left[\int_t^T e^{-r(s-t)}mG\,ds + \int_t^T e^{-r(s-t)}\lambda G E^{\mathbb{Q}}[v_s]\,ds\right]{}_{T-t}p_{x+t} \\
&= \left[\int_t^T G(m+\lambda\theta)e^{-r(s-t)}ds + \int_t^T \lambda G(v_t-\theta)e^{-(r+\kappa)(s-t)}ds\right]{}_{T-t}p_{x+t} \\
&= G\left[(m+\lambda\theta)\left(\frac{1-e^{-r(T-t)}}{r}\right)+\lambda(v_t-\theta)\left(\frac{1-e^{-(r+\kappa)(T-t)}}{r+\kappa}\right)\right]{}_{T-t}p_{x+t}
\end{aligned}$$

(e) Explain whether you agree or disagree with the following statements made by your analyst.

(i) "The new rider fee is not a function of A_t, therefore it is not sensitive to changes in the account value."

(ii) "The new rider fee has a positive Vega."

Commentary on Question:
Candidates performed poorly in this part. For those that attempted the question, most candidates did not recognize that the account value and the rider fee are correlated.

(i) Disagree. As account value decreases, v_t will tend to increase due to $\rho < 0$, and therefore rider fee will increase.

(ii) Agree. The vega of the expected PV of the new rider fee is positive. As $\sqrt{v_t}$ increases, the rider fee increases.